Habilitation Thesis Reviewer's Report

Masaryk University	
Faculty	Faculty of Science
Procedure field	Mathematics - Geometry
Applicant	Mgr. Vojtěch Žádník, Ph.D.
Applicant's home unit, institution	Masaryk University
Habilitation thesis	Geometric constructions and correspondences in action
Reviewer	Paweł Nurowski, prof. Dr hab.
Reviewer's home unit, institution	Center for Theoretical Physics, Polish Academy of Sciences

The habilitation thesis of Dr. Vojtech Zadnik concerns with two very active and fashionable areas of Differential Geometry. These are: Cartan Geometry, and its special incarnation, Parabolic Geometry. This second area, now vividly developing over the World, has its craddle in the Central Europe, actually in Austria and the Czech Republic. It got the initial speed by the works of Great Masters, such as Elie Cartan, Shiing-shen Chern, Israel Bernstein and Bertram Kostant, and was codified through the fundamental works of professors Andreas Čap from Vienna and Jan Slovak from Brno, and their collaborators. Dr Vojtech Zadnik's scietific interests, as interests of a former student and a close collaborator of prof. Slovak, are naturally related to this area of mathematics.

As mentioned before, although the center of the Parabolic Geometry activity is in the Central Europe, it is developed in many countries in the World, in prestigious academic centers. I only mention those that immediately come to my mind: Great Britain (D. Calderbank, Bath; M. Dunajski, Cambridge; P. Tod, Oxford), Ireland (B. McKay, Cork), France (O. Bicquard, J. Merker, Paris), Germany (H. Baum, Berlin; V. Matveev, Jena; Th. Metler, Frankfurt), Italy (G. Mano, Torino; A. Medvedev, Trieste), Poland (B. Jakubczyk, W. Kryński, P. Nurowski, Warsaw; A. Borówka, Kraków), Norway (B. Kruglikov, D. The, Tromso), Belarus (B. Doubrov), Japan (Th. Morimoto, Kyoto; K. Yamaguhi, Sapporo), Australia (M. Cowling, Sydney; M. Eastwood, Th. Leistner, Adelaide), New Zealand (R. Gover, Auckland), and the USA (R. Bryant, C, Robbles, Durham; Ch. Feferman, Princeton; J.M. Landsberg, I. Zelenko, College Station; R. Graham, Seattle).

So what is a Parabolic Geometry for a nonspecialist?

It is a geometric structure that evolved from the classical geometries such as *projective* and *conformal* geometries. By focusing on the relevant properties of these two geometries, and on a little less known but still classical, Cauchy-Riemann (CR) geometry, one abstracts a notion - the *parabolic geometry* - that generalizes these three cases. After this abstraction, some other well known (but forgotten) geometries, such as for example the geometry of generic rank 2 distributions in dimension 5, or the geometry of 3rd order ODEs considered modulo contact transformation of variables, can be understood as other examples of this realm. More importantly, as a result of this abstraction an aboundance of new geometries appears. And this happens *even in low dimensions*!

This opens plenty new areas of studies. In particular, grouping a(n infinite) number of geometries together, one finds many similarities between them. This enables to translate a theorem, or a construction, from one parabolic geometry to the other. Surprisingly this can produce *new* results/constructions even in the classical cases of conformal and projective geometries. Simply, when a theory of a given geometry has been developed it was driven by problems in its geometric setting. Understanding a theorem/construction as a parabolic geometry theorem/construction, in, say, conformal geometry raises a question what is its counterpart, say, in the CR geometry. Since these geometries were developed independently, it may happen that the developers in the one area were not seing this what independent developers saw in the other.

I made this (a bit long) introduction, to be able to say that dr Zadnik's thesis is a perfect illustration of the methodology I mentioned in the last few lines.

The thesis describes the original results from four published papers.

The first of these papers concerns with the geometry of *chains.* This notion was introduced by Elie Cartan in the 1930ties (years before the term 'parabolic geometry' was invented) in case of the geometry of real hypersurfaces in $\mathbb{C}^2$. When a hypersurface in $\mathbb{C}^2$ is the sphere, $\mathbb{S}^3 = \{\mathbb{C} \ni (z,w) : |z|^2 + |w|^2 = 1\}$, chains arise as the intersections of the sphere with complex lines $\ell(\lambda) = \lambda(z_0, w_0)$. As seen in this example, at every point of the sphere $\mathbb{S}^3$, which is a particular case of a *Levi nondegenerate* 3-dimensional *CR manifold*, there exists precisely one chain in every direction. This equips this CR manifold, the $\mathbb{S}^3 \subset \mathbb{C}^2$, with the so called *path geometry* - the geometry of a family of curves on a manifold having the property that through its every point passes precisely one curve in each direction. The main point here is that both, the *geometry of CR manifolds* (of hypersurface type and Levi nondegenerate), as well as the *path geometry*, are examples of a *parabolic geometry.* Thus, also the term *chain* must have its parabolic interpretation, and can be generalized from the Cartan's

3-dimensional CR case to other parabolic geometries.

This is the background for the first set of results of Dr Zadnik. These are presented in Section 4 of his habilitation thesis. They come from his paper [17], joint with A. Čap, which is published in the *top* mathematical journal *J. Diff. Geom.* Based on the fact that a hypersurface type Levi-nondegenerate CR manifolds belong to the class of parabolic *contact* geometries, dr Zadnik in Ref. [17], generalizes Cartan's chains to parabolic *contact* geometries other than CR. As in the CR case, the generalized chains on its own define an associated parabolic geometry, which is the *path geometry of chains*. The main concern of the paper is if, and when, one can reconstruct the original parabolic contact geometry from the path geometry of its chains. It is shown that it can be done when the original parabolic contact geometry is CR (as in the Cartan's and, more generally Chern-Moser's case) or in the integrable Lagrangean case. In other parabolic contact geometry cases the integrabilty conditions for this reconstruction are so strong that it can only be done in the flat cases. These are very interesting results.

Another set of dr Zadnik's habilitation results is about the conformal Patterson-Walker metrics. These results are described in Section 5 of his text, and are originally published in his coauthored papers [43] and [44]. Here I describe them briefly:

Given a torsion-free affine connection ∇ on an n-dimensional manifold M, it was shown by Patterson and Walker that any such connection determines a natural metric g_∇ of signature (n, n) on the cotangent bundle T^*M. Likewise, if instead one is given a projective class $[\nabla]$ of torsion-free connections $[\nabla]$, i.e., and equivalence class of connections sharing the same unparametrized geodesics, it determines a natural conformal class of metrics $[g_\nabla]$ on T^*M. Dr Zadnik, with his collaborators, studies these conformal structures. He calls them conformal Patterson-Walker structures.

The construction and characterization of these conformal structures is presented in two ways. The first approach is to study the construction as a generalized Fefferman construction. The classical Fefferman construction of a conformal structure on a circle bundle over a CR manifold was generalized by Andreas Čap to a *generalized-Fefferman-construction-between-parabolic-geometries.* The work of dr Zadnik and his collaborators follows Čap's general treatment of constructions of this type. Local Cartan geometric constructions of essentially the same type, but not using the framework introduced by Čap, have appeared in earlier work of mine and George Sparling in dimension $n = 2$. Its higher-dimensional case was also discussed in another paper of mine ["Projective versus metric structures", *J. Geom. Phys.* 62 (2012) 657-674, DOI 10.1016/j.geomphys.2011.04.011], preceeding the papers of dr Zadnik's team.

In my opinion, an important part of the research in parabolic geometry should focus on applications of the available general constructions (as e.g. codified in Čap - Slovak's book) to interesting geometric structures. The work of dr Zadnik and his collaborators in papers [42]-[44] fulfills this requirement. It presents a detailed picture of an interesting instance of a construction between parabolic geometries. Also it deals with difficult technical questions, such as the normality of the induced Cartan connection.

The main new results presented in [42]-[43] concern with a characterization of the resulting conformal structures in terms of conformally invariant data defined on the conformal manifold. In the second part of their work dr Zadnik and his collaborators use a more direct approach to study Conformal Patterson-Walker metrics. They also present results about symmetries of the constructed conformal structures and results concerning the question when the conformal class formed by the Patterson-Walker metric contains Einstein metrics.

Their work culminates in their third paper [44], which, in my opinion, is the nicest part of their study. The authors provide a very neat geometric construction of an explicit ambient metric for the Patterson-Walker conformal structures discussed in their earlier works. I stress that there are very few known cases, when the Fefferman-Graham ambient construction can be performed explicitely. I consider it as a kind of a miracle, that in the conformal Paterson-Walker case investigated by dr Zadnik and his collaborators, the ambient metric can be obtained explicitely (I mean with the explicit formulae) in a purely geometric way.

The last part of dr Zadnik's habilitation thesis is a report on his results from the paper [62] describing the theory of conformal curves. After giving a very nice historical summary of a geometry of curves in various geometric settings (the beginning of Section 6 of the thesis), dr Zadnik shows his way of introducing a kind of the Frenet frame to a curve, a frame that captures its *conformal* properties. This enables for a construction of conformal invariants of the curve. The results are obtained in a collaboration with Josef Silhan. They produce the desired conformal Frenet frame by means of *tractors*.

I like this part of the mathematical contribution of dr Zadnik very much and I rate it very high. Historically, the theory of curves and their invariants gave the first instance of differential invariants of geometric objects. Thus the subject developed by Zadnik and Silhan is placed in the very roots of the differential geometry. And nevertheless dr Zadnik with his collaborator produces here impressive new results!

Summarizing I want to say that the mathematics used and developed by dr Zadnik is of high quality. All his four papers and his original results described in Chapter II of his habilitation thesis are important contributions to (a) parabolic geometries, (b) Feferman-Graham ambient theory and (c) conformal geometry. They show that he is an active mathematician working on interesting problems. I strongly support his application for the habilitation.

Reviewer's questions for the habilitation thesis defence

Q: The Fefferman like conformal structures for n-dimensional projective geometries were considered in my paper 'Projective versus metric structures', *J. Geom. Phys.* 62 (2012) 657-674, DOI 10.1016/j.geomphys.2011.04.011. These were obtained following directly constructions described in my paper with Sparling about the 2-dimensional projective case, and in my paper about the conformal metrics for the $(2,3,5)$ distributions. If the dimension of the projective structure (M,p) is n, it is shown on page 662, Theorem 1.8, of the paper 'Projective versus metric structures', that there exists a number n of conformal Fefferman like structures over (M,p), each of them having the signature (n,n) and each defined by the projective data from (M,p). In case of $n=2$ the construction from the 'Projective versus metric structures' paper gives two conformal structures, which modulo being selfdual, or anti-selfdual, are the same. In this case they coincide with the conformal metric obtained in the join paper with Sparling (of course, in the special case when my construction with Sparling is restricted from the path geometric case to the the projective case).

The thesis of dr Zadnik does not quote my paper 'Projective versus metric structures' in the context of the conformal Patterson-Walker metrics being Fefferman like structures (actually it does not quote it at all). Does this mean that if $n > 2$ none of my conformal structures from Theorem 1.8, on p. 662, is conformal Patterson-Walker? I have the opposite feelling. Actually I think that all n conformal structures I introduce there, are conformally equivalent to the conformal Patterson-Walker metrics dr Zadnik talks about. Can he clarify this point?

Conclusion

The habilitation thesis entitled "Geometric constructions and correspondences in action" by Mgr. Vojtěch Žádník, Ph.D. ***fulfils*** requirements expected of a habilitation thesis in the field of Mathematics - Geometry.

Warszawa 5.03.2020 Paweł Nurowski