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Annex No. 11 to the MU Directive on Habilitation Procedures and Professor Appointment Procedures

HABILITATION THESIS REVIEWER'S REPORT

Masaryk University	
Faculty	Faculty of Sciences
Procedure field	Mathematics - Geometry
Applicant	Mgr. Vojtěch, Žádník, Ph.D.
Applicant's home unit, institution	Faculty of Education and Faculty of Science, Masaryk University, Brno
Habilitation thesis	Geometric constructions and correspondences in actions
Reviewer	Professor Michel Eastwood, BA, MA, PhD, FAA
Reviewer's home unit, institution	The University of Adelaide

Please see the following four pages for my detailed review.

Report on

Geometric constructions and correspondences in action

by Vojtěch Žádník

Habilitation Thesis at Masaryk University

This thesis follows the standard format with an original extended introduction of approximately 60 pages, followed by reproductions of some published articles, in this case the following four articles:

1. A. Čap and V. Žádník, *On the geometry of chains*, Jour. Diff. Geom. **82** (2009) 1–33.
2. M. Hammerl, K. Sagerschnig, J. Šilhan, A. Taghavi-Chabert, and V. Žádník, *A projective-to-conformal Fefferman-type construction*, SIGMA **13** (2017), Paper No. 081, 33 pp.
3. M. Hammerl, K. Sagerschnig, J. Šilhan, A. Taghavi-Chabert, and V. Žádník, *Fefferman-Graham ambient metrics of Patterson-Walker metrics*, Bull. Lond. Math. Soc. **50** (2018) 316–320.
4. J. Šilhan and V. Žádník, *Conformal theory of curves with tractors*, Jour. Math. Anal. Appl. **473** (2019) 112–140.

Regarding the style of this thesis, the writing in the extended introduction is absolutely superb and a joy to read. There are very few misprints and the grammar is almost perfect. Indeed, it is written with a certain amount of wit that carries the reader along. It is clear from this part of the thesis that the author has a firm grasp on the detail and generalities concerning the Cartan/parabolic geometries that are the subject of this thesis. I would have no hesitation in recommending this part of the thesis to a student wanting to enter this fascinating and internationally thriving field. I would also be comfortable if Vojtěch Žádník were supervising such a student. He clearly has the overview and insight that such a task requires.

Regarding his four published articles chosen for this thesis, my assessment is less favourable. Let me describe them individually. Article [1] is one of three published articles written with Andreas Čap (one of which also includes his PhD supervisor, Jan Slovák). They are all concerned with 'distinguished curves' in parabolic geometry, either in generality or in various special cases of 'chains' in a parabolic contact geometry. Article [1] is concerned with the important special case of chains in hypersurface CR structures. All of these articles are excellent and well-cited contributions to the field. Distinguished curves are absolutely fundamental in parabolic geometry and these articles (in addition to Žádník's 2003 PhD thesis) are foundational. I would also like to draw your attention to

• B.M. Doubrov and V. Žádník, *Equations and symmetries of generalized geodesics*, in Differential Geometry and its Applications, Matfyzpress 2005, pp. 203–216.

Unfortunately, this article is difficult to obtain and therefore little known. In my opinion, however, it is a seminal contribution. The authors directly relate, by dint of the Cartan connection, *unparameterised* distinguished curves in a parabolic geometry to the *symmetry algebra* of homogeneous curves in the flat model. Whilst distinguished curves come equipped with a family of preferred parameterisations, it is surely a good idea to separate these two phenomena so that one can investigate where these curves go and how they are parameterised independently. Even in the seemingly simple case of conformal geometry this separation is illuminating.

Articles [2] and [3] should be discussed together since they are concerned with the same construction. The 'Patterson-Walker construction' canonically associates to any torsion-free affine connection on an n-dimensional smooth manifold M, a neutral signature metric on the cotangent bundle T^*M viewed as a $2n$-dimensional manifold. This construction is nearly 70 years old,

• E.M. Patterson and A.G. Walker, *Riemann extensions*, Quart. Jour. Math. **3** (1952) 19–28,

and from a modern perspective is easy to state: one uses the connection to split the tangent bundle of T^*M into *vertical* and *horizontal* subspaces which are decreed to be *null* for the metric we seek and we complete this with the canonical pairing between horizontal and vertical vectors tautologically identified with tangent and co-tangent vectors down on M. This construction has been discussed by many authors since that time. However, an early observation by Walker,

• A.G. Walker, *Riemann extensions of non-Riemannian spaces*, Convegno Internazionale di Geometria Differenziale, Venezia 1953, Edizioni Cremonese 1954, pp. 64–70,

that the neutral signature metric on T^*M is conformally flat if and only if the original connection is projectively flat, seems to have gone unnoticed. In any case, this is a strong clue that the original construction should be modified so that a projective class of connections on M gives rise to a neutral signature conformal metric. This can be done by tensoring T^*M with a suitable power of $\det(TM)$ pulled back to T^*M. After some erroneous attempts in the literature, the correct formulation is given in [2]. Having the correct formulation, the authors of [2] and [3], then go on to develop this story in a fairly straightforward manner. For example, since the *Fefferman-Graham ambient metric construction*, is somewhat degenerate in the projective setting, one might expect an ambient metric construction to all orders for the corresponding neutral signature metric. The authors check that this is, indeed, the case. But this is as it should be: once the initial construction has been correctly formulated, articles [2] and [3] are the natural and straightforward continuation with no surprises. In my opinion, the correct formulation is the main contribution. Unfortunately, this formulation is already in the much earlier and unpublished

- M. Hammerl and K. Sagerschnig, *A non-normal Fefferman-type construction of split-signature conformal structures admitting twistor spinors*, arXiv:1109.4231,

from 2011 by two of the five authors of [2] and [3]. Additionally, it goes without saying that [2] and [3] have five authors and that Žádník's contribution is therefore diluted. A good sole author article would, of course, be more convincing.

Finally, there is the article [4] by Šilhan and Žádník. The aim of this article is to 'present the general theory of curves in conformal geometry using tractor calculus' (taken from the abstract). Already, the distinguished curves in conformal geometry, often known as 'conformal circles' since they are circles on the round sphere, and the conformally invariant 'preferred parameterisations' on general curves have been presented using tractors. Therefore, one might expect some higher invariants of generic curves to be developed in this article. This is, indeed, the case, being nicely accomplished by a sort of 'Frenet frame' for tractors. One might also expect some application of these invariants and Section 3.2 begins with the statement

> *As an application of the current approach we describe some conserved quantities (or first integrals) along conformal circles for some specific conformal structures.*

However, there seem to be no 'specific conformal structures' at all in this article beyond flat space and even the flat case is not much explained. This is in stark contrast with the excellent (but uncited in [4]) article

• K.P. Tod, *Some examples of the behaviour of conformal geodesics*, Jour. Geom. Phys. **62** (2012) 1778–1792,

which is just bulging with interesting examples. Sadly, article [4] has no punchline. This is despite there being many open and important questions around conformal circles. Whether they can 'spiral' in general is still unknown. Regarding a specific conformal structure, where do the circles go on $\mathbb{CP}_2$ with its Fubini-Study conformal metric? In summary, article [4] is distinctly lacking in motivation and is devoid of applications.

Nevertheless, on the basis of this thesis I have no hesitation is recommending that Vojtěch Žádník be awarded Habilitation status in the field of Mathematics & Geometry at Masaryk University. Although his recent published work is not so strong, his earlier work including [1] is very good indeed and the original extended introduction to this Habilitation thesis is truly excellent.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)
I have no questions for the defence

Conclusion

The habilitation thesis entitled “Geometric constructions and correspondences in actions” by Vojtěch Žádník **fulfils** the requirements expected of a habilitation thesis in the field of Mathematics - Geometry.

Date: 6th March 2020 Signature: