MASARYKOVA UNIVERZITA

Annex No. 11 to the MU Directive on Habilitation Procedures and Professor Appointment Procedures

Habilitation Thesis Reviewer's Report

Masaryk University	
Faculty	Faculty of Science
Procedure field	Mathematics - Geometry
Applicant	Mgr. Vojtěch Žádník, Ph.D.
Applicant's home unit, institution	Masaryk University
Habilitation thesis	Geometric constructions and correspondences in action
Reviewer	Prof A. Rod Gover, DPhil (Oxford), MSc, BSc Hons (Canterbury).
Reviewer's home unit, institution	University of Auckland, New Zealand

Following your request, I am writing to comment on the Habilitation Thesis of Dr. Vojtech Zadnik. Dr Zadnik works in the area of mathematics known as differential geometry and is especially expert in the use of an approach known as the Cartan geometry approach to treat and understand structures that are important for geometry and that are also important for the study of mathematical physics and geometric partial and ordinary differential equations, as well as the links of these to other areas such as integrable systems and complex analysis. I am a full professor in mathematics expert in differential geometry in similar areas, so I am in a good position to comment on the candidate's research.

The candidate is previously known to me as a professional colleague, we have not collaborated. I am basing my assessment on the information provided in the Habilitation Thesis as well as the resources that are available via the internet (such as MathSciNet and similar). It should be noted that this area of mathematics is one where both the publication rate and citation rate is low compared to other disciplines. This is in part culture, but also because there is a large degree of learning involved in each different direction pursued.

First, as general context, Dr Zadnik is regularly publishing in journals that range from very good to really excellent, including the *Journal of Differential Geometry, Bull. Lond. Math. Soc., J. Math. Anal. Appl., and Geom. Dedicata.* These journals have a very high rejection rate and only accept articles that are original, accurate, deep, and of significant interest. Thus the fact that the applicant has a number of such articles may be regarded as immediate evidence of very strong research. For example, the *Journal of Differential Geometry* is not only the absolutely top pure differential geometry journal but is among the top of all mathematics journals according to impact (e.g. using MathSciNet data).

The Thesis structure involves two Chapters and an Appendix. The main chapter is the second which consists of the treatment of the core material where the author has made major contributions. This develops and summarises in a uniform manner a body of work due to the

author and his collaborators, as mainly published in four articles that are reproduced in the Appendix. The first Chapter concerns general background.

The introductory Chapter develops the tools needed for the specific geometric problems treated in the later sections. It concerns Cartan geometries, which are a very broad class of continuously varying geometric structures that can be understood using a rather general framework -- and this latter feature is one of the really key aspects of the approach of the Thesis. Among the general ideas are those of the Cartan connection and similar/equivalent objects such the tractor connection. The Habilitation provides an excellent brief and unifying account of these key tools. Also given is how the Cartan picture allows a really beautiful and practically useful treatment of certain ``extensions'' of geometries and the related notions of correspondence spaces that historically have been of great importance for the treatment of integral transforms (such as the X-ray transform and the Penrose transform). The next part of the background reduces to the special case of parabolic geometries. This is still a vast class of structures, but one that admits a very strong uniform treatment using the Cartan machinery and its refinements, as there described. Some important notions are then touched on in this setting such those of distinguished curves and BGG sequences -- both directions where the author has made contributions (and in fact major contributions in the case of curves). Finally in the background the focus is reduced further to look at examples of broad classes of concrete geometries that fit into the more general picture. These parts nicely illustrate the general procedure in a concrete setting while also laying more detailed foundations for the later sections.

The main Chapter, rather understatedly called ``Usage'', is composed of 3 sections: the Geometry of Chains, Conformal Patterson-Walker metrics, and the Conformal theory of curves. These are three quite different directions where the author has contributed new major advances.

Chains are distinguished curves in the class of parabolic geometries known as parabolic contact structures, and are special curves that are nowhere tangent to the contact distribution. They are of considerable historical importance following the work of Cartan, Chern and Moser, and others, on what is known as the equivalence problem. The focus of the development here, following the candidates's work with Cap in reference [17], is on the description of a "weaker" underlying geometry, associated with the space of chains, that is called a path geometry. This step is helpful for the conceptual understanding to the role of chains in geometry. A key question attacked is whether it is possible to recover original the Cartan geometry from the underlying path geometry using the general extension theory that is developed in Section 1.3 of the article. In the candidate's (joint) work it is shown that for the classes of structures known as almost CR and Lagrangian this works completely if and only if the geometry satisfies a certain integralibity condition (the so-called "torsion-free" condition) which agrees with the "usual" integrability condition that is well known in the case of CR structures. On the other hand for more general parabolic contact structures it is shown that the full procedure works only in the flat case, and it is important to have this scope established. Another valuable output of the work is having the different possible cases put in, and treated, in a uniform conceptually simple framework.

Conformal Patterson-Walker metrics are another class of geometric structures that arise as an extension from a simpler geometry. They are of considerable current interest. The original

idea of Walker and Patterson-Walker (see e.g. reference [54]) concerns split signature Riemannian metrics constructed from general affine connections in half the dimension, but the idea was extended somewhat in the literature as explained carefully in the Thesis. In collaborative work the author and collaborators (see references [29,42,43,44]) develop and extend this idea, describe it as a functorial construction, and also in details how it links it to important developments due to Fefferman-Graham, Nurowski-Sparling, and Dunajski-Tod. In particular interpreting and extending the work of Dunajski-Tod, it is shown that a slight modification of the original Walker construction, of metrics on the contangent bundle over affine manifolds, yields the result that projectively equivalent affince connections give conformally invariant metrics. A key theme brought out in the unifying development is that the entire programme fits nicely as a subcase of the general Cartan extension theory from the background chapters and in particular the idea of ``Fefferman type constructions" as described in references [13,15] and elsewhere. This has valuable concrete applications -- for example it is explained that one is able to immediately conclude that that Bach tensor and Fefferman-Graham obstruction tensor (a key conformal invariant) must vanish for any Patterson-Walker metric in the conformal class. It is also shown that another invariant, the so-called Q curvature (of Branson) must vanish for any metric in the conformal class. This is quite surprising as the Q-curvature is not conformally invariant and perhaps suggests that some interesting further theory might be available.

The third section of Chapter II treats distinguished curves in conformal geometry. This covers first a summary of related historical developments that are central to the area, and in particular the classical idea of Frenet frames and associated invariants along curves, and then how these are generalised by the Cartan moving frame. This, along with the known tractor calculus treatment and characterisation of conformal curves, is then the basis for the development of a conformally invariant Frenet Frame approach that is developed following the author's article [62] with Silhan. A main application is an effective and systematic way to treat the construction of absolute and relative conformal invariants, that is, conformal invariants of the curve which are independent of parameterisation or, respectively, which transform in a simple way under change of parameterization, respectively. Distinguished conformally invariant parametrisations for curves are also treated and form part of the theory. This work with Silhan is really quite an impressive development and I am certain will be the basis of much further work by researchers worldwide.

In summary Dr Zadnik works in broadly applicable area of differential geometry that interacts deeply with several other areas of mathematics. He is very well recognised for his research work, and has excellent qualitative and quantitative (relative to the field norms) indicators of impact and scholarship. The body of work presented in the Habilitation thesis, as well as the attachments and other related articles by Dr Zadnik, show considerable breadth of expertise as well as depth and in my opinion clearly demonstrate internationally leading research excellence. They also show that he is on a very good trajectory for further important advances and contributions.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)

Q1. On page 49 it is explained that the Q-curvature of a Patterson–Walker metric vanishes. This implies that the kernel of the critical GJMS operators is rather large for such metrics and should have a special role. Has the candidate thought about this? In particular is there a there a secondary invariant available along the lines of the Q'-curvature from CR geometry?

Q2. There will be a Frenet theory for chains and the other distinguished curves of CR geometry – an analogous theory to the conformal theory. The existence of this is touched on in the Thesis but, to what extent is this now understood by the candidate. Are there new subtleties?

Conclusion

The habilitation thesis entitled "Geometric constructions and correspondences in action" by Mgr. Vojtěch Žádník, Ph.D. ***fulfils*** the requirements expected of a habilitation thesis in the field of Mathematics - Geometry.

25.2. 2020

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A Rod Gover, FRSNZ