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PARAMETRISATION METHODS FOR CONSTRUCTIVE ANALYSIS OF BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Habilitation Thesis

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Abstract. This thesis is a commented collection of the papers [35, 36, 49, 58, 60–63] dealing with the same topic, namely, the constructive analysis of various boundary value problems for systems of ordinary differential equations. The methods under consideration are constructive in the sense that they allow both to obtain approximate solutions and to use them in order to prove the solvability of the problem.

The approach is based on Lyapunov-Schmidt type reduction. Systems of ordinary differential equations on compact intervals are considered.

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Preface.

The systematic investigation of differential and integral equations rewritten in the form of an abstract equation

$$Lu = Nu \tag{1}$$

with non-invertible linear L dates back to Lyapunov and Schmidt; detailed historical remarks and references can be found in the book [12]. The idea to convert (1), by using suitable projectors, into a system of two equations one of which is easier to investigate and the other is of lower dimension (usually finite), is referred to as the Lyapunov-Schmidt method. The finite-dimensional part is then called the determining, or branching equation. The ways of diversification of the method reflect different approaches to the construction of (1) and to obtaining the determining equations and their subsequent study. For the periodic and more general boundary value problem for ordinary differential equations similar ideas, in different forms and under different assumptions, had led to numerous results (see, e.g., the books [16, 27, 65, 72, 74] and references therein). A common assumption of the majority of works is a certain smallness of the non-linearity, which, in relevant cases, leads close to the averaging method [6].

An interesting method of this kind, which is applicable for a wide enough class of equations with "large" non-linearities, is due to Cesari ([9]; see also [24, 67, 78]); his approach uses approximations of the Galerkin type. The important feature there is the rare ability to prove the existence of an exact solution based on the constructed approximation. A close approach was suggested by Hale for the study of periodic solutions [16]; both are sometimes referred to as the Cesari-Hale method (e.g., [26]). Different techniques based on parametrisation, which also belong to this group and likewise allow to obtain solvability conditions, were developed in [55, 65, 72]; they originate from Samoilenko's works [69, 70]. These approaches arose in relation to the periodic problem and have a common feature in the idea of perturbing the equation appropriately in order to eliminate secular terms. The topic of the present thesis belongs to this group of methods.

The thesis is a commented collection of the papers [35, 36, 49, 58, 60–63] dealing with the constructive investigation of boundary value problems for systems of ordinary differential equations. The approach is constructive in the sense that it allows both to explicitly obtain approximate solutions and to use the results of computation in the solvability analysis. It is essentially different from the Cesari method and, in our opinion, has certain advantages (in particular, a simpler idea is used to guarantee the convergence for large non-linearities under rather mild assumptions). The existence theorems involve conditions that use the properties of approximations and are verified directly.

Our aim here is the extension and development of the techniques from [55, 65] in several directions. We suggest a new approach allowing one to treat boundary value problems for systems with large non-linearities using a convergent iteration procedure and thus overcome the commonly assumed smallness conditions. At the same time, effective conditions sufficient for the solvability can be formulated and approximations of solutions obtained. The techniques are rather flexible and, as is shown, can be adopted for application to a wide range of problems.

We restrict ourselves to systems of ordinary differential equations on compact intervals, the main assumption is that the non-linearity is Lipschitzian in a bounded domain. The periodic, two-point and general non-local boundary conditions are considered.

A commented review of the papers is contained in Chapters I ([60, 61]), II ([58, 62, 63]), III ([36, 49]), IV ([35]). The numbering of sections, propositions and equations is continuous; the notation is somewhat modified for better readability and does not always coincide with that used in the papers. The references to the papers and their short summary are given below, their offprints start at p. 40.

[60] A. Rontó, M. Rontó and N. Shchobak. 'Constructive analysis of periodic solutions with interval halving'. *Bound. Value Probl.* (2013). 2013:57, 1–34.

[61] A. Rontó, M. Rontó and N. Shchobak. 'Notes on interval halving procedure for periodic and two-point problems'. *Bound. Value Probl.* (2014). 2014:164, 1–20.

[63] A. Rontó, M. Rontó and J. Varha. 'A new approach to non-local boundary value problems for ordinary differential systems'. *Appl. Math. Comput.* 250 (2015), 689–700.

[58] A. Rontó, M. Rontó and J. Varha. 'On non-linear boundary value problems and parametrisation at multiple nodes'. *Electron. J. Qual. Theory Differ. Equ.* (2016). Paper No. 80, 1–18.

[36] I. Rachůnková, L. Rachůnek, A. Rontó and M. Rontó. 'A constructive approach to boundary value problems with state-dependent impulses'. *Appl. Math. Comput.* 274 (2016), 726–744.

[49] A. Rontó, I. Rachůnková, M. Rontó and L. Rachůnek. 'Investigation of solutions of state-dependent multi-impulsive boundary value problems'. *Georgian Math. J.* 24:2 (2017), 287–312.

[35] B. Půža, A. Rontó, M. Rontó and N. Shchobak. 'On solutions of nonlinear boundary-value problems the components of which vanish at certain points'. *Ukrain. Math. J.* 70:1 (2018), 101–123.

[62] A. Rontó, M. Rontó and N. Shchobak. 'Parametrisation for boundary value problems with transcendental non-linearities using polynomial interpolation'. *Electron. J. Qual. Theory Differ. Equ.* (2018). Paper No. 59, 1–22.

The works [60, 61] concern the development and extension of the method of periodic successive approximations (see [55]). Systems of ordinary differential equations with non-linearities satisfying the Lipschitz condition in a bounded domain are considered under the periodic and other two-point boundary conditions. We suggest a constructive approach to the study of such boundary value problems which is applicable without restrictions on the magnitude of the Lipschitz constant (it may therefore be arbitrarily large). Effective conditions guaranteeing the existence of a periodic solution are given and a relation to theorems on topologic continuation is clarified.

In [58, 63] we show how the parametrisation techniques developed for two-point problems can be used in the case of the problem with a general non-local boundary condition (in particular, using multiple parametrisation nodes in a general position).

In [36], we develop parametrisation techniques for the investigation of solutions of two-point boundary value problems for systems of ordinary differential equations with state-dependent jumps. In such systems, not only the value of a jump but also the time instant when it should occur depends on the current value of the solution and is determined from whether the trajectory touches a certain surface. This is a relatively new and little studied class of systems (see, e. g., [37]), for which there are almost no effective methods for construction of solutions (§ 11). The character of the problem leads one to the use of parametrisation techniques in a natural way and [36] shows that a method suitable for this case can be suggested along the lines of [58, 61]. We study solutions having only one jump in the given time interval; the boundary condition is linear two-point. The work [49] is a continuation of [36] and deals with the case where a solution may have multiple jumps in the given time interval; the boundary condition is non-linear two-point. A scheme of analysis of such problems is formulated and examples of its practical realisation are presented.

Principles used in [36, 49, 58, 61] can also be used for other problems, in particular, for the problem on a solution of a non-linear ordinary differential equation vanishing at a certain point the value of which is not known *a priori*. In [35], we study a system of non-linear first order ordinary differential equations under two-point boundary conditions and suggest a method for the analysis of solutions whose components vanish at certain unknown points. We consider the case where every component of the solution vanishes only once on the given interval.

In [62], we describe and justify a convenient version of the method from [63] where the polynomial interpolation over Chebyshev nodes is used. This version is suitable for systems satisfying additional regularity conditions in the time variable and has the advantage of significanty easier construction of approximations in practice.

Notation and symbols.

Let $n \in \mathbb{N}$, $G \subset \mathbb{R}^n$ be a closed bounded set, and $[a, b]$ be a bounded interval. The following symbols are used in the sequel:

1. $\mathbf{1}_n$ is the unit matrix of dimension n.
2. $r(K)$ is the maximal, in modulus, eigenvalue of a matrix K.
3. For any vector $x = (x_k)_{k=1}^n$, we write $|x| := \sum_{k=1}^n |x_k| e_k$, where e_k, $k = 1, 2, \ldots, n$, are the respective columns of $\mathbf{1}_n$.
4. If $\{x, y\} \subset \mathbb{R}^n$, we write $x \le y$ (resp., $x < y$) if and only if $x_k \le y_k$ (resp., $x_k < y_k$) for all $k = 1, 2, \ldots, n$.
5. If $h = (h_i)_{i=1}^n : Q \to \mathbb{R}^n$ is a continuous function, where $Q \subset \mathbb{R}^m$ is closed, $m \le n$, we write
$$\max_{z \in Q} h(z) := \sum_{k=1}^n \max_{z \in Q} h_k(z) e_k, \quad \min_{z \in Q} h(z) := \sum_{k=1}^n \min_{z \in Q} h_k(z) e_k. \tag{2}$$
6. Given a closed interval $J \subseteq [a, b]$, we put
$$\delta_{J,G}(f) := \max_{(t,z) \in J \times G} f(t, z) - \min_{(t,z) \in J \times G} f(t, z) \tag{3}$$
in the sense of (2). When $J = [a, b]$, we omit the interval and write
$$\delta_G(f) := \delta_{[a,b],G}(f). \tag{4}$$
7. For any $z \in \mathbb{R}^n$ and $\varrho \in \mathbb{R}^n_+$, we write
$$\mathscr{O}_\varrho(z) := \{\xi \in \mathbb{R}^n : |z - \xi| \le \varrho\} \tag{5}$$
with the componentwise definition of $|\cdot|$ and $\le$. For $G \subset \mathbb{R}^n$, we put
$$\mathscr{O}_\varrho(G) := \bigcup_{\xi \in G} \mathscr{O}_\varrho(\xi). \tag{6}$$
8. If $D \subset \mathbb{R}^n$ and $\varrho \in \mathbb{R}^n_+$, we put
$$D(\varrho) := \{z \in D : \mathscr{O}_\varrho(z) \subset D\}, \tag{7}$$
where $\mathscr{O}_\varrho(z)$ is given by (5).
9. ∂G is the boundary of a set G.
10. $\mathscr{C}(G_0, G_1)$: see Definition II.1, p. 16.
11. $\rhd_S$: see Definition I.5, p. 9.

12. If K is a square matrix of dimension n, $\mathrm{Lip}_K(G)$ stands for the set of functions $g : G \to \mathbb{R}^n$ satisfying the componentwise Lipschitz condition

$$|g(z_1) - g(z_2)| \leq K|z_1 - z_2| \tag{8}$$

for all z_1 and z_2 from G. We use the same notation $f \in \mathrm{Lip}_K(G)$ if $f : [a, b] \times G \to \mathbb{R}^n$ and $f(t, \cdot) \in \mathrm{Lip}_K(G)$ for all or a. e. $t \in [a, b]$.

13. If $a \leq t_0 < t_1 \leq b$, we put $\alpha_0(t; t_0, t_1) := 1$ and

$$\alpha_m(t; t_0, t_1) = \left(1 - \frac{t - t_0}{t_1 - t_0}\right) \int_{t_0}^{t} \alpha_{m-1}(s; t_0, t_1) ds + \frac{t - t_0}{t_1 - t_0} \int_{t}^{t_1} \alpha_{m-1}(s; t_0, t_1) ds \tag{9}$$

for all $t \in [t_0, t_1]$ and $m = 1, 2, \ldots$. If $t_0 = a$ and $t_1 = b$, we omit the last two arguments an write $\alpha_m(\cdot)$ instead of $\alpha_m(\cdot; a, b)$.

14. H_k^β, where $k \in \mathbb{R}_+^n$, $k_i \geq 0$, $0 < \beta_i \leq 1$, $i = 1, 2, \ldots, n$, is the set of vector functions $y : [a, b] \to \mathbb{R}^n$ satisfying the Hölder conditions

$$|y_i(t) - y_i(s)| \leq k_i |t - s|^{\beta_i} \tag{10}$$

for all $\{t, s\} \subset [a, b]$, $i = 1, 2, \ldots, n$.

I. Successive approximations for periodic problems.

The chapter is based on [60, 61].

In [60, 61], we develop a numerical-analytic approach to the analysis of periodic solutions of systems of non-autonomous ordinary differential equations using an idea from [54]. The method is numerical-analytic in the sense that its realisation consists of two stages concerning, respectively, an explicit construction of certain equations and their numerical analysis, and is close in the spirit to the Lyapunov–Schmidt reductions [12] (however, neither small parameter nor implicit function argument is used).

We focus on numerical-analytic schemes based upon successive approximations. In the context of the theory of non-linear oscillations, such types of methods were apparently first developed in [8, 16, 69, 70] (we refer to [44, 46, 47, 51, 53–55, 65, 66, 72, 73] for the related bibliography), at first for the periodic problem

$$u'(t) = f(t, u(t)), \qquad t \in [0, p], \tag{11}$$

$$u(0) = u(p). \tag{12}$$

Compared with other methods, they are interesting, among the rest, by the ease of application and the possibility to establish the solvability of the problem along with obtaining its approximate solutions (see also §§ 5, 11).

The numerical-analytic approach replaces the boundary value problem by a family of auxiliary problems containing certain free parameters (in particular, for (11), (12) these are the Cauchy problems and the parameter has the meaning of the initial value of the solution). The solution of the auxiliary problem is sought for in an analytic form by successive approximations, whereas the numerical value of the parameter is determined later from the so-called determining equations. In order to guarantee the convergence, a kind of the Lipschitz condition is usually assumed [55, 65, 72, 73] and a smallness restriction of the type

$$r(K) \le \frac{c}{p} \tag{13}$$

is imposed, where K is the Lipschitz matrix, p is the length of the interval, and c is a constant. An improvement of condition (13) consists in maximising the value of c.

Conditions of form (13) are expectable in this context and cannot be dropped: if (13) is violated, then the applicability of the method is not guaranteed. A natural question arises how to make the method work in such cases. In [60], we suggest a constructive approach to the investigation of the periodic problem (11), (12) using the interval halving technique which, with rather general assumptions, allows us to construct a similar scheme convergent under the weaker condition

$$r(K) \le \frac{2c}{p}. \tag{14}$$

The scheme of [60] can be used, in particular, in the cases where the conditions of the works [55, 69–72] do not hold. The restriction imposed on the width of the domain is likewise relaxed, which, together with (14), has its effect on the conditions sufficient for the solvability (§ 2). This leads one to a method which is applicable without smallness conditions of type (13) (§ 4).

In its original form (see, e. g., [65]), the numerical-analytic approach to (11), (12) suggests to look for periodic solutions of (11) among the limit functions of the n-parametric family of function sequences given by the recurrence relation

$$u_m(t,\xi) := \xi + \int_0^t f(s, u_{m-1}(s,\xi))ds - \frac{t}{p}\int_0^p f(s, u_{m-1}(s,\xi))ds, \quad t \in [0,p], \tag{15}$$

where $m \geq 1$, $u_0(t,\xi) := \xi$, $t \in [0,p]$, and ξ is a vector parameter. Under the Lipschitz condition $f \in \mathrm{Lip}_K(D)$ on a suitable domain D, the existence of the limit $u_\infty(\cdot,\xi) := \lim_{m\to\infty} u_m(\cdot,\xi)$ is established for all ξ from a certain $D_0 \subset D$ (provided that such a set exists). Then the existence of a solution $u(\cdot)$ of the periodic problem (11), (12) with the value at zero lying in D_0 can be interpreted in terms of the solvability of the equation

$$\int_0^p f(s, u_\infty(s,\xi))ds = 0$$

with respect to ξ. This leads one to a Lyuapunov–Schmidt type reduction of the periodic problem, the applicability of which is guaranteed by the smallness assumptions (13) (see [55, 60] for more details). The set D where f is Lipschitzian should also be sufficiently wide so that the existence of a non-empty D_0 is guaranteed,[1] namely,

$$D\left(\frac{p}{4}\delta_D(f)\right) \neq \varnothing, \tag{16}$$

where $\delta_D(f) := \max\left\{\delta_{[0,p/2],D}(f),\ \delta_{[p/2,p],D}(f)\right\}$ with $\delta_{[0,p/2],D}(f)$ and $\delta_{[p/2,p],D}(f)$ defined as indicated in notation 6, p. 1. The set in (16) is defined according to (7) (notation 8, p. 1).

In order to guarantee the convergence, conditions of type (13) (i. e., a certain smallness of the eigenvalues of the matrix pK) are assumed. It is proved in [65] that the method using sequence (15) is applicable under the condition

$$r(K) < \frac{1}{\gamma_0 p}, \tag{17}$$

where

$$\gamma_0 := \frac{3}{10} \tag{18}$$

(references concerning different values of γ_0 in (17) can be found in [55]). In the cases where (17) does not hold the method, generally speaking, cannot be used.

In [60], we show that this limitation can be overcome by noticing that the quantity which is assumed to be small enough is always proportional to the length of the interval. A natural interval halving technique then allows one to produce a version of the scheme where (17) is replaced by the condition

$$r(K) < \frac{2}{\gamma_0 p}$$

and, thus, weakened by half. A similar improvement is also achieved in relation to condition (16), which is replaced by the assumption that

$$D\left(\frac{p}{8}\delta_D(f)\right) \neq \varnothing. \tag{19}$$

[1] In particular, such that $\operatorname{diam} D \geq \frac{p}{2}\delta_D(f)$, with the componentwise definition of a vector-valued diameter of a set.

It is clear that the transition to (19) weakens (16) by half and, in principle, this construction allows one to apply the modified method without imposing restrictions of type (17) (at the expense of more computational work; see § 8).

In [61], we improve the scheme of [60] so that its substantiation is simplified and, in particular, replace (16) by an assumption which is more transparent and, generally speaking, less restrictive.[2] The setting from [61] simplifies formulations and allows one to drop certain technical details (in particular, there is no more need to verify unpleasant conditions of type (16) and (19)[3]). This also makes it particularly easy to adopt the approach to certain other problems (see also § 6).[4]

§ 1. Periodic successive approximations on half-intervals. Consider the periodic boundary value problem (11), (12), where $p \in (0, \infty)$, $f : [0, p] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Carathéodory conditions, and a solution is an absolutely continuous vector function satisfying (11) almost everywhere on $[0, p]$.

Let us fix a closed bounded set $G \subset \mathbb{R}^n$, where the initial values of solutions of problem (11), (12) will be looked for. We choose G to be convex.

We suppose that the non-linearity f in (11) is Lipschitzian in G with respect to the space variable: there exist a certain matrix K such that[5]

$$f \in \operatorname{Lip}_K(G). \tag{20}$$

Let ξ and η be arbitrary vectors from G. Let us put

$$x_0(t, \xi, \eta) := \left(1 - \frac{2t}{p}\right) \xi + \frac{2t}{p} \eta, \qquad t \in [0, p/2], \tag{21}$$

$$y_0(t, \xi, \eta) := 2 \left(1 - \frac{t}{p}\right) \eta + \left(\frac{2t}{p} - 1\right) \xi, \qquad t \in [p/2, p], \tag{22}$$

and define the recurrence sequences of functions $x_m : [0, p/2] \times G^2 \to \mathbb{R}^n$ and $y_m : [p/2, p] \times G^2 \to \mathbb{R}^n$, $m = 0, 1, \ldots$, according to the formulae

$$\begin{aligned} x_m(t, \xi, \eta) := x_0(t, \xi, \eta) &+ \int_0^t f(s, x_{m-1}(s, \xi, \eta))\, ds \\ &- \frac{2t}{p} \int_0^{\frac{p}{2}} f(s, x_{m-1}(s, \xi, \eta))\, ds, \quad t \in [0, p/2], \end{aligned} \tag{23}$$

$$\begin{aligned} y_m(t, \xi, \eta) := y_0(t, \xi, \eta) &+ \int_{\frac{p}{2}}^t f(s, y_{m-1}(s, \xi, \eta))\, ds \\ &- \left(\frac{2t}{p} - 1\right) \int_{\frac{p}{2}}^p f(s, y_{m-1}(s, \xi, \eta))\, ds, \quad t \in [p/2, p], \end{aligned} \tag{24}$$

[2] In [60] and all the previous works, one fixes a set D where the non-linearity is known to be Lipschitzian and then checks its subsets $D(\varrho)$ of form (7) that can potentially contain initial values of periodic solutions. This is somewhat unnatural because, in any case, it is the initial values that are of major interest, the regularity assumptions for the equation being only technical conditions induced by the method of proof. Instead, it is more logical to choose [61] a closed bounded set $G \subset \mathbb{R}^n$, where one expects to find initial values of the solution, and to assume that the non-linearity is Lipschitzian on a suitable $\tilde{G} \supset G$, with $\tilde{G}$ only as large as the method requires. The argument of [60] then leads us to the choice $\tilde{G} = G_\varrho$, where $G_\varrho := \mathscr{O}_\varrho(G)$ is the ϱ-neighbourhood of G with a suitable ϱ.

[3] The practical verification of (16), (19) is inconvenient, and in some cases it is not so easy to construct a suitable D. In contrast to this, [61] no more contains assumptions of type (19), while conditions (35), (36) given below are verified directly (and usually can be checked with the help of software; see, e. g., the discussion in § 15 in the context of a problem on vanishing solutions).

[4] Besides its more natural character, the use of the pair of sets (G_ϱ, G) is also advantageous in contrast to $(D, D(\varrho))$ because, geometrically, $D(\varrho)$ does not necessarily copy the shape of D (see pp. 77, 78).

[5] See notation 12, p. 2.

where $m \geq 0$. Equalities (21), (22) mean that we choose $x_0(\cdot, \xi, \eta)$ and $y_0(\cdot, \xi, \eta)$ as linear functions on the appropriate intervals satisfying the equalities $x_0(0, \xi, \eta) = \xi$, $x_0(p/2, \xi, \eta) = \eta$, $y_0(p/2, \xi, \eta) = \eta$, $y_0(p, \xi, \eta) = \xi$.[6] The idea of the approach is based on the following statement.

Proposition I.1. *Let $(\xi, \eta) \in G^2$ be fixed. If the limits $x_\infty(\cdot, \xi, \eta)$ and $y_\infty(\cdot, \xi, \eta)$ of sequences* (23) *and* (24)*, respectively, exist uniformly on $[0, p/2]$ and $[p/2, p]$, then:*

1. *The function $x_\infty(\cdot, \xi, \eta)$ has the property $x(p/2) - x(0) = \eta - \xi$ and is the unique solution of the problem*

$$x'(t) = f(t, x(t)) + \frac{2}{p}\Xi(\xi, \eta), \qquad t \in [0, p/2], \tag{25}$$

$$x(0) = \xi, \tag{26}$$

where

$$\Xi(\xi, \eta) := \eta - \xi - \int_0^{\frac{p}{2}} f(\tau, x_\infty(\tau, \xi, \eta))d\tau. \tag{27}$$

2. *The function $y_\infty(\cdot, \xi, \eta)$ has the property $y(p) - y(p/2) = \xi - \eta$ and is the unique solution of the problem*

$$y'(t) = f(t, y(t)) + \frac{2}{p}\mathrm{H}(\xi, \eta), \qquad t \in [p/2, p], \tag{28}$$

$$y\Big(\frac{p}{2}\Big) = \eta, \tag{29}$$

where

$$\mathrm{H}(\xi, \eta) := \xi - \eta - \int_{\frac{p}{2}}^{p} f(\tau, y_\infty(\tau, \xi, \eta))d\tau. \tag{30}$$

In such cases, solutions of problem (11), (12) can be looked for in the form $u_\infty(\cdot, \xi, \eta)$, where $u_\infty(\cdot, \xi, \eta) : [0, p] \to \mathbb{R}^n$ is defined by setting

$$u_\infty(t, \xi, \eta) := \begin{cases} x_\infty(t, \xi, \eta) & \text{if } t \in [0, p/2], \\ y_\infty(t, \xi, \eta) & \text{if } t \in (p/2, p], \end{cases} \tag{31}$$

for all ξ and η from G. The next theorem gives conditions ensuring that $x_\infty(\cdot, \xi, \eta)$ and $y_\infty(\cdot, \xi, \eta)$ are well defined for $(\xi, \eta) \in G^2$ and formula (31) makes sense (in particular, function (31) is continuous on $[0, p]$ for any $(\xi, \eta) \in G^2$).

The formulations of the theorems below require the knowledge of the Lipschitz constant of f on a set somewhat wider than G. More precisely, we require that the Lipschitz condition, which is known to hold on G, is also satisfied on its certain componentwise neighbourhood $G_\varrho := \mathscr{O}_\varrho(G)$, and select K so that, instead of (20),

$$f \in \mathrm{Lip}_K(G_\varrho) \tag{32}$$

with a certain ϱ the value of which depends on the amplitude of values of f.

Introduce the functions[7]

$$\bar{\alpha}_1(t) := \frac{2}{p}t(p - t), \qquad t \in [0, p/2], \tag{33}$$

$$\bar{\bar{\alpha}}_1(t) := \frac{1}{2p}(p - 2t)(2t - 3p), \qquad t \in [p/2, p]. \tag{34}$$

[6] The construction can also be derived by a “gluing” argument for certain auxiliary problems [60].

[7] In fact, $\bar{\alpha}_1(t) = \alpha_1(t; 0, p/2)$ and $\bar{\bar{\alpha}}_1(t) = \alpha_1(t; p/2, p)$ for all $t \in [0, p]$, where α_1 is given by (9), p. 2.

Theorem I.2. *If there exists a non-negative vector ϱ with the property*

$$\varrho \geq \frac{p}{8}\delta_{G_\varrho}(f) \tag{35}$$

such that (32) *holds with a certain K and*[8]

$$r(K) < \frac{2}{\gamma_0 p}, \tag{36}$$

then, for all fixed $(\xi,\eta) \in G^2$, the sequence $\{x_m(\cdot,\xi,\eta) : m \geq 0\}$ (resp., $\{y_m(\cdot,\xi,\eta) : m \geq 0\}$) converges to a limit function $x_\infty(\cdot,\xi,\eta)$ (resp., $y_\infty(\cdot,\xi,\eta)$) uniformly in $t \in [0,p/2]$ (resp., $t \in [p/2,p]$), and the following estimates hold:

$$|x_m(\cdot,\xi,\eta) - x_\infty(t,\xi,\eta)| \leq \frac{\bar{\alpha}_1(t)}{2^{m+1}} (\gamma_0 pK)^m \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 pK\right)^{-1} \delta_{[0,p/2],G_\varrho}(f) \tag{37}$$

for all $t \in [0,p/2]$ and

$$|y_m(\cdot,\xi,\eta) - y_\infty(t,\xi,\eta)| \leq \frac{\bar{\bar{\alpha}}_1(t)}{2^{m+1}} (\gamma_0 pK)^m \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 pK\right)^{-1} \delta_{[p/2,p],G_\varrho}(f) \tag{38}$$

for all $t \in [p/2,p]$ and $m \geq 3$.

Assumption (36), which, by Theorem I.2, ensures the applicability of the iteration scheme based on formulae (23), (24), is twice as weak as assumption (17) for the original sequence (15). The same kind of imrovement is achieved concerning the condition on the set where f is Lipschitzian: for the scheme without interval halving, one would require that

$$\exists \varrho : \quad \varrho \geq \frac{p}{4}\delta_{G_\varrho}(f), \tag{39}$$

which, compared to (35), is twice as strong. In contrast to the related assumptions from [60] and the earlier works, condition (35) is easier to verify because in order to do so one has only to find the value $\delta_{G_\varrho}(f)$, which is computed directly according to formula (3). In addition, it is usually possible to estimate this value by using the properties of particular non-linear terms.

Theorem I.3. *Assume that* (32) *holds, where ϱ is a vector with property* (35) *and K satisfies condition* (36). *Then, for every solution $u(\cdot)$ of problem* (11), (12) *with the property*

$$\{u(t) \mid t \in [0,p]\} \subset G_\varrho \text{ and } \{u(0), u(p/2)\} \subset G, \tag{40}$$

there exists a pair (ξ_0,η_0) in G^2 such that $u(\cdot) = u_\infty(\cdot,\xi_0,\eta_0)$. On the other hand, the function $u_\infty(\cdot,\xi,\eta)$ is a solution of the periodic boundary value problem (11), (12) *if and only if the pair (ξ,η) satisfies the system of $2n$ equations*[9]

$$\Xi(\xi,\eta) = 0, \quad \mathrm{H}(\xi,\eta) = 0. \tag{41}$$

Theorem I.3 provides a formal reduction of the periodic problem (11), (12) to the system of $2n$ numerical equations (41) in the sense that the initial data $(u(0), u(p/2))$ of any solution of (11), (12) with properties (40) can be found from (41). Thus, under the conditions assumed, the question on solutions of the periodic boundary value problem (11), (12) can be replaced by that of the system of numerical equations (41).

[8] Condition (36) can be slightly improved by replacing γ_0 by the constant $\gamma_* \approx 0.2927$ appearing independently in several works (see [55, p. 585] for references). The inconvenience this causes is that the corresponding analogues of estimates (37), (38) are guaranteed for m sufficiently large only, which affects the proof of Theorem I.6, while the effect of the refinement is insignificant (in fact, $\gamma_0 - \gamma_* \approx 0.0073$). Furthermore, the aim of the interval division here is to weaken assumption (17) to (36).

[9] In view of Theorem I.2, the functions $\Xi : G^2 \to \mathbb{R}^n$ and $\mathrm{H} : G^2 \to \mathbb{R}^n$ are well defined by relations (27) and (30).

§ 2. Solvability analysis. The argument based on Theorems I.2 and I.3 implies a scheme of investigation of the periodic problem (11), (12) which allows one both to construct approximate solutions and to establish the solvability of the problem in a rigorous way. This approach is realised using the approximate determining functions

$$\Xi_m(\xi,\eta) := \eta - \xi - \int_0^{p/2} f(\tau, x_m(\tau,\xi,\eta))d\tau, \tag{42}$$

$$\mathrm{H}_m(\xi,\eta) := \xi - \eta - \int_{p/2}^{p} f(\tau, y_m(\tau,\xi,\eta))d\tau \tag{43}$$

considered for a fixed value of m and, thus, computable explicitly. Then, as in [60], the function

$$u_m(t,\xi,\eta) := \begin{cases} x_m(t,\xi,\eta) & \text{if } t \in [0,p/2], \\ y_m(t,\xi,\eta) & \text{if } t \in (p/2,p]. \end{cases} \tag{44}$$

can be used to obtain the mth approximation to a solution of problem (11), (12) provided that we are able to find certain roots (ξ,η) of the mth approximate determining equations

$$\Xi_m(\xi,\eta) = 0, \quad \mathrm{H}_m(\xi,\eta) = 0. \tag{45}$$

System (45) plays the role of an approximate version of (41) and, in contrast to (41), all the terms involved in (45) can be computed in a finite number of steps.

Under suitable additional conditions, the existence of solutions of the periodic problem (11), (12) can be derived from the solvability of system (45). To proceed, choose a closed region G in $\mathbb{R}^n$ and put

$$\Phi_m(\xi,\eta) := \begin{pmatrix} \eta - \xi - \dfrac{1}{2}\displaystyle\int_0^p f\left(\frac{p-\tau}{2}, x_m\left(\frac{p-\tau}{2},\xi,\eta\right)\right) d\tau \\ \xi - \eta - \dfrac{1}{2}\displaystyle\int_0^p f\left(\frac{p+\tau}{2}, y_m\left(\frac{p+\tau}{2},\xi,\eta\right)\right) d\tau \end{pmatrix} \tag{46}$$

and

$$\Phi_\infty(\xi,\eta) := \begin{pmatrix} \eta - \xi - \dfrac{1}{2}\displaystyle\int_0^p f\left(\frac{p-\tau}{2}, x_\infty\left(\frac{p-\tau}{2},\xi,\eta\right)\right) d\tau \\ \xi - \eta - \dfrac{1}{2}\displaystyle\int_0^p f\left(\frac{p+\tau}{2}, y_\infty\left(\frac{p+\tau}{2},\xi,\eta\right)\right) d\tau \end{pmatrix} \tag{47}$$

for any $(\xi,\eta) \in G^2$.

Theorem I.4 ([61]). *Let* (32) *hold, where* ϱ *is a certain vector with property* (35) *and* K *satisfies condition* (36). *Moreover, assume that* Φ_m *does not vanish on* ∂G^2 *and satisfies the condition*[10]

$$\deg(\Phi_m, G) \neq 0 \tag{48}$$

for a certain fixed $m \geq 0$ *and there exists a continuous mapping* $Q : [0,1] \times G^2 \to \mathbb{R}^{2n}$ *which does not vanish on* $(0,1) \times \partial G^2$ *and is such that*

$$Q(0,\cdot) = \Phi_m, \qquad Q(1,\cdot) = \Phi_\infty.$$

Then there exists a pair $(\xi^*,\eta^*) \in G^2$ *such that the function* $u := u_\infty(\cdot,\xi^*,\eta^*)$ *is a solution of the periodic boundary value problem* (11), (12) *possessing properties* (40).

[10] The vector field Φ_m is finite-dimensional and the degree involved in (48) is the Brouwer degree [12], $\deg(\Phi_m, G) = \deg_B(\Phi_m, G, 0)$.

Let the binary relation $\rhd_S$ be defined for any $S \subset \mathbb{R}^r$, $r \geq 1$, as follows.[11]

Definition I.5 ([55])**.** Functions $g = (g_i)_{i=1}^l : \mathbb{R}^r \to \mathbb{R}^l$ and $h = (h_i)_{i=1}^l : \mathbb{R}^r \to \mathbb{R}^l$, $l \geq 1$, are said to satisfy the relation $g \rhd_S h$ if and only if there exists a function $\nu : S \to \{1, 2, \ldots, l\}$ such that $g_{\nu(z)}(z) > h_{\nu(z)}(z)$ at every point $z \in S$.

Theorem I.6 ([61])**.** *Let $f \in \mathrm{Lip}_K(G_\varrho)$, where ϱ satisfies inequality* (35) *and K has property* (36)*. Let, moreover,*

$$|\Phi_m| \rhd_{\partial G} \frac{5p}{18} \begin{pmatrix} M_m \delta_{[0,p/2],G}(f) \\ M_m \delta_{[p/2,p],G}(f) \end{pmatrix} \tag{49}$$

for a certain fixed $m \geq 2$, where

$$M_m := \left(\frac{\gamma_0 p}{2}\right)^{m+1} K^{m+1} \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 p K\right)^{-1}. \tag{50}$$

Then there exists a pair $(\xi^*, \eta^*) \in G^2$ such that $u := u_\infty(\cdot, \xi^*, \eta^*)$ is a solution of problem (11), (12) *possessing properties* (40).

Note that conditions of Corollary I.6 are assumed for a fixed m, and all the values depending on it are evaluated in finitely many steps. When the order of iteration is growing, it is clear from (36) and (50) that

$$\lim_{m \to \infty} M_m = 0$$

and, hence, the right-hand side of inequality (49) vanishes when m grows to $+\infty$. On the other hand, it is easy to see that, under the conditions assumed, the mapping Φ_m (uniformly on compact sets) converges to Φ as m tends to $+\infty$. We thus arrive at the interesting observation that assumption (49) of Theorem I.6, which is the main condition ensuring the non-degeneracy of the homotopy, has the form of the strict inequality

$$|\Phi_m| \rhd_{\partial G} w_m, \tag{51}$$

where $|\Phi_m|$ tends to $|\Phi|$ while the term w_m becomes arbitrarily small as m grows to $+\infty$. In this way, with the growth of the number of iteration, m, the corresponding condition (49) gradually becomes less and less restrictive (which, of course, does not imply its eventual fulfilment: if the problem under consideration is not solvable, then the condition will never be satisfied).

Under conditions of Theorems I.4 and I.6, the system of $2n$ numerical equations (45) has at least one solution $(\xi^{[m]}, \eta^{[m]})$ in G^2, the periodic problem (11), (12) has a solution u, and the function[12]

$$U_m(t) := u_m(t, \xi^{[m]}, \eta^{[m]}), \qquad t \in [0, p], \tag{52}$$

defined according to (44) can be regarded as an approximation (mth approximation) of u. This is justified, in particular, by the estimates

$$|x_m(\cdot, \xi^{[m]}, \eta^{[m]}) - U_m(t)| \leq \frac{\bar{\alpha}_1(t)}{2^{m+1}} (\gamma_0 p K)^m \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 p K\right)^{-1} \delta_{[0,p/2],G_\varrho}(f)$$

[11] The binary relation $\rhd_S$ introduced by Definition I.5 is a kind of strict inequality for vector functions and its properties are similar to those of the usual strict inequality sign. For example, $f \geq g$ and $g \rhd_S h$ imply that $f \rhd_S h$ for any non-epmpty S.

[12] The meaning of the variables whose values appearing in (52) are determined from equations (45) is quite clear: $\xi^{[m]}$ is an approximation of the initial value of the p-periodic solution u at 0 and $\eta^{[m]}$ is that of $u(p/2)$.

for all $t \in [0, p/2]$ and

$$|y_m(\cdot, \xi^{[m]}, \eta^{[m]}) - U_m(t)| \leq \frac{\bar{\bar{\alpha}}_1(t)}{2^{m+1}} (\gamma_0 pK)^m \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 pK\right)^{-1} \delta_{[p/2,p],G_\varrho}(f)$$

for $t \in [p/2, p]$ and $m \geq 3$. Further inequalities can be obtained by estimating the mapping $(\xi, \eta) \mapsto u_m(t, \xi, \eta)$ for $t \in [0, p]$ fixed.

In this way, an approximate solution is obtained, in a sense, automatically once the solvability of the problem has been proved. On the other hand, the proof of solvability itself uses properties of finitely many iterations, i. e., the results of computation turn out to be helpful for obtaining a qualitative result. The analysis of problem (11), (12) along these lines is constructive, the assumptions are verified directly.

When computing higher iterations in order to obtain approximate solutions, it is helpful to apply suitable simplified versions of the algorithm which are better adopted for use with suitable software (see § 3).

Theorem I.7. *For $m = 1$, the assertion of Theorem I.6 holds with* (50) *replaced by the condition*

$$|\Phi_1| \rhd_{\partial G} \frac{p^3}{144} \begin{pmatrix} K^2 \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 pK\right)^{-1} \delta_{[0,p/2],G}(f) \\ K^2 \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 pK\right)^{-1} \delta_{[p/2,p],G}(f) \end{pmatrix}. \tag{53}$$

Meaningful results are also obtained in the case of zeroth approximation, i. e., when no iterations are carried out at all. Note that the zeroth approximation itself is rather rough: the periodic solution is approximated by a piecewise linear function (see Figure 3, p. 67, and Figure 6, p. 88); its construction is very easy. The analysis of the zeroth approximation is, however, rather useful since it gives us certain preliminary information on the solution. In particular, this helps us to choose the domains in a more optimal way (not too large) and, thus, avoid unnecessary computations on sets where we do not expect to find solutions. This is noticeable, in particular, when computing the values $\delta_{[0,p/2],G}(f)$ and $\delta_{[p/2,p],G}(f)$ according to formula (3).

With the given function f involved in equation (11), we asociate the function $f^{\#} : G^2 \to \mathbb{R}^{2n}$ by putting

$$f^{\#}(\xi, \eta) := \begin{pmatrix} \eta - \xi - \dfrac{1}{2}\displaystyle\int_0^p f\left(\frac{p-\tau}{2}, \frac{\tau}{p}\xi + \left(1 - \frac{\tau}{p}\right)\eta\right) d\tau \\ \xi - \eta - \dfrac{1}{2}\displaystyle\int_0^p f\left(\frac{p+\tau}{2}, \frac{\tau}{p}\xi + \left(1 - \frac{\tau}{p}\right)\eta\right) d\tau \end{pmatrix} \tag{54}$$

for any $(\xi, \eta) \in G^2$.

Theorem I.8. *Assume that there is a certain vector ϱ with property* (40) *and $f \in \mathrm{Lip}_K(G_\varrho)$ with K satisfying inequality* (36)*. Let, furthermore,*

$$\deg(f^{\#}, G) \neq 0 \tag{55}$$

and

$$|f^{\#}| \rhd_{\partial G} \frac{5p^2}{108} \begin{pmatrix} K \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 pK\right)^{-1} \delta_{[0,p/2],G}(f) \\ K \left(\mathbf{1}_n - \frac{1}{2}\gamma_0 pK\right)^{-1} \delta_{[p/2,p],G}(f) \end{pmatrix}. \tag{56}$$

Then the p-periodic problem (11), (12) *has at least one solution $u(\cdot)$ which possesses properties* (40).

The function $f^{\#}$ involved in Theorem I.8 can be regarded as an analogue of the *averaged* map

$$\bar{f}(\xi) := \int_0^p f(s,\xi)ds \tag{57}$$

for $\xi \in G$, which corresponds to the scheme without interval halving:

Corollary I.9. *Let $f \in \mathrm{Lip}_K(G_\varrho)$ with ϱ satisfying* (39) *and K such that* (17) *holds. If*

$$\deg(\bar{f}, G) \neq 0 \tag{58}$$

and

$$|\bar{f}| \rhd_{\partial G} \frac{5p^2}{27} K (\mathbf{1}_n - \gamma_0 pK)^{-1} \delta_{G_\varrho}(f), \tag{59}$$

then the p-periodic problem (11), (12) *has a solution $u(\cdot)$ with properties* (40).

We see that conditions (40), (36) of Theorem I.8 are twice as weak as the corresponding conditions (39), (17) of Corollary I.9. One may also note the values of constants on the right-hand side of (56) and (59) (see (51)).

Assumption (58) with $\bar{f}$ given by (57) arises frequently in topological continuation theorems where the homotopy to the averaged equation is considered (see, e. g., [27]). Arguing in this manner, we can obtain, in particular, the assertion of the well-known Mawhin's theorem [27] for the Lipschitzian case considered here. In this context, Theorem I.8 can be regarded as its "halved" analogue where the system of equations

$$f^{\#}(\xi, \eta) = 0 \tag{60}$$

determines the initial data of the zeroth approximation. We see that the interval division, as a result of which the p-periodic solution is constructed by gluing together two curves, leads one to the presence of two independent variables, ξ and η, due to which system (60), in contrast to the case without interval division,

$$\int_0^p f(t,\xi)dt = 0, \tag{61}$$

contains n extra equations. Equation (60) can be regarded as a kind of analogue of (61), which is an equation arising in a natural way in asymptotic methods [6].

§ 3. Approximation scheme in practice. A practical analysis of the periodic problem (11), (12) starts directly with the computation of iterations. This is preferable because, before verifying the conditions, it is useful to get, by carrying out numerical experiments, a preliminary guess of the initial values. Numerical tests also help one to choose the domain in a reasonable way and avoid checking the conditions in regions where roots are unlikely to be found.

We construct the approximate determining equations (45), solve them numerically in an appropriate region, substitute the corresponding roots into the formula for u_m, and form functions (52) which are, in a sense, candidates for approximations of a solution. Having constructed functions (52) for several values of m, we check their behaviour heuristically and if it shows enough signs of convergence, we stop the computation and verify the assumptions of the existence theorem. In the case of success, we conclude that the existence of a solution is guaranteed in the region we work with, and either we are satisfied with the achieved accuracy of the approximation (in this case, the scheme stops and the function U_m given by (52) for the last computed value of m is considered as its outcome) or we carry out one more step to improve it and perform a similar check again. This leads one to the algorithm given in § 4.

It is interesting to observe that, once the existence of a solution is known from Theorem I.4 at the mth step of iteration, we immediately obtain an approximation to it in form (52). The scheme thus allows us to both study the solvability of the periodic problem and construct approximations to its solution. On the other hand, the solvability analysis based on Theorem I.4 uses the results of computation: conditions involve the expressions obtained from the iteration formulae (23), (24) and the domain, in which the conditions are verified, is selected after numerical solution of one or several determining systems.

It should be noted that the ability to derive the fact of solvability of the original problem from the corresponding properties of approximate problems is rather uncommon (see [55] for some details). For the numerical methods, the generic situation is, in fact, quite the reverse, when some or another technique is applied to solve a problem which is *a priori* assumed to be solvable.

The most difficult part of the scheme consists in the analytic construction of functions (42), (43) with m so large as is sufficient to establish the solvability of the periodic problem using Theorem I.4 and achieve the required precision of approximation. Its practical implementation, for which symbolic computation systems such as MAPLE and MATHEMATICA are of much help, can be considerably facilitated by combining the analytic computation with a suitable kind of approximation. The use of the polynomial [59, 62] or trigonometric [73] interpolation is very convenient for this purpose (see also § 10). Another useful modification is the "reuse" of computed values when passing to the next step of iteration [49].

§ 4. Repeated interval halving. An algorithm. The interval halving procedure can be repeated, in which case the conditions are weakened by half at each step.

A scheme with multiple interval halvings is constructed similarly to § 1. The interval halvings cause the growth of the dimension of the determining system, which contains $2^k n$ equations at the kth interval division. One can regard this as a certain price to be paid for being able to convert a possibly divergent scheme into a convergent one. For a discussion of two other approaches to this issue, see § 5.

For the scheme with multiple interval halvings, the initial approximation depends on more parameters, but its construction is also very easy: the graph of $u_0(\cdot, \xi)$ is a broken line joining the corresponding nodes.[13] Once $u_0(\cdot, \xi)$ is constructed, the formulas for the subsequent approximations are derived automatically by rescaling the projection map to the corresponding subintervals.

Repeated interval halvings allow one to suggest the following algorithm [60, 61] of investigation of the periodic problem (11), (12).

Algorithm I.10.

1. Fix a certain k_0 and consider the scheme with k_0 interval divisions. Fix an m_0 and construct $u_m(\cdot, \xi)$ for $m = 0, 1, \dots m_0$.
2. Solve the mth approximate determining equations for ξ, find a root $\xi^{[m]}$, and put

$$U_m(t) := u_m(t, \xi^{[m]}), \qquad t \in [0, p],\ m = 0, 1, \dots m_0. \tag{62}$$

   In case the equation has multiple roots, the related analysis is repeated for each of them (one can study multiple solutions of the original problem in this way).[14]

3. Check the behaviour of the functions U_0, U_1, …, U_{m_0} constructed according to (62) (the heuristic step). If there are noticeable signs of convergence, choose a

[13] See Figure 6, p. 88.

[14] We assume that the roots are isolated.

suitable[15] G containing the graph of U_{m_0}, find a ϱ from the condition

$$\varrho \geq \frac{p}{2^{k_0+2}} \delta_{G_\varrho}(f), \tag{63}$$

compute the Lipschitz matrix K for f in G_ϱ, and verify the convergence condition

$$r(K) < \frac{2^{k_0}}{\gamma_0 p}. \tag{64}$$

If not successful with either (63) or (64), increase k_0 appropriately and try again. If (63) and (64) are both satisfied, proceed to step 4.

4. Verify conditions of the existence theorem for G and m_0. If not satisfied, or if the accuracy of U_{m_0} is insufficient, pass to $m = m_0 + 1$ and study U_{m_0+1}. Otherwise the algorithm stops, and the outcome is:

   (a) There is a solution u of (11), (12), and $u \approx U_{m_0}$;

   (b) $\exists(\xi_*, \eta_*) \in G^2$: $u(\cdot) = u_\infty(\cdot, \xi_*, \eta_*)$;

   (c) The space localisation of the graph of u is described by properties (40).

Note the role of interval divisions in the algorithm: for K not satisfying the smallness condition (17) and $k_0 = 0$ (i. e., when u_m is constructed according to (15) without any interval divisions), the algorithm would stop at step 3. However, it is obvious that (63)[16] holds if k_0 is chosen to be large enough.

Moreover, the sequence of numbers

$$\varrho^{(k)} := \inf\Big\{ \varrho : \varrho \geq \frac{p}{2^{k+2}} \delta_{G_\varrho}(f) \Big\}, \quad k = 1, 2, \ldots, \tag{65}$$

is obviously monotone decreasing to 0, and therefore $\{G_{\varrho^{(k)}} : k \geq 1\}$ is a decreasing sequence of sets tending to the original G as k grows. The Lipschitz condition (32) with ϱ chosen from sequence (65) thus approaches the original assumption (20). Clearly, (64) is always satisfied for suitable k_0.

In other words, if we put $\varrho = \varrho^{(k_0)}$ with k_0 large enough, it follows that the method under consideration is theoretically applicable however large the eigenvalues of K may be.

§ 5. Remarks. The proposed technique has features which can be regarded as certain advantages over other approaches. For example, when applying it, one experiences no difficulties with the selection of the starting approximation (in contrast, e. g., to monotone iterative methods); there is no need to re-calculate considerable amounts of data when passing to the next step of approximation (unlike projection methods); the global Lipschitz condition and the existence, uniqueness and extendability of solutions of the Cauchy problems are not assumed (unlike IVP, or shooting methods).

The form of the smallness conditions assumed for the non-linearity implies that the scheme is appropriate, in particular, for the study of high-frequency oscillations. This corresponds to problems on small time intervals, in which case no divisions are needed.

[15] Before coming to this point, it is advisable to start by checking the zeroth approximation, i.e., consider the related formulas with $m = 0$. At this point, no iteration is carried out at all, and all the expressions are computed explicitly. The zeroth approximation is easy to construct and it provides us with a preliminary information on a possible choice of the domain G. This observation is applicable to all the versions of the approach corresponding to different situations.

[16] More precisely, the condition on the existence of a non-negative vector ϱ such that (63) holds.

By carrying out some amount of computation analytically, one partly avoids accumulation of rounding errors that would typically arise when operating with very small or large numbers. From this point of view, the interval divisions may additionally serve the purpose of reducing computational errors (in particular, in the cases where the solution depends sensitively on the initial values; see also the discussion in [75], § 7.3.4).

In the scheme with multiple interval divisions (§ § 4, 9), the determining equations involve variables representing the values of the solution at multiple points; the approach can be therefore regarded as an efficient alternative to multiple shooting [1, 30, 75] which may be well applicable also in the cases where the shooting methods cannot be used. The last may happen either due to the complicated character of the system (multiple shooting does not work, e. g., for systems with jumps at unknown times, [3, p. 3]), the type of boundary conditions[17] or the failure to satisfy the necessary assumptions (usually one assumes a sufficient smoothness of the non-linearity to guarantee the applicability of Newton type methods, which are commonly used to solve the resulting equations for the initial values; see, e. g., [75, p. 516] and [76, p. 375]).[18] Furthermore, in order to apply shooting methods, it is necessary to ensure that the initial value problem for the given differential equation has always a unique solution which is extendable to the entire time interval under consideration. The smoothness of the non-linearity alone, as is well-known, is insufficient for this: e. g., for the equation $u' = \frac{1}{2}u^3$ the solution of the initial value problem $u(0) = c$ has the form

$$u(t) = \frac{c}{\sqrt{1 - c^2 t}}$$

and is undefined for $t \geq 1/c^2$. The construction we develop here assumes only the Lipschitz condition on a bounded set and uses auxiliary initial value problems for which the unique solvability and extendability of the solution are guaranteed.

It should be noted in general that, when applying numerical methods, the existence of a solution is assumed *a priori*. In contrast to this, our approach allows one also to prove the solvability of the problem in a rigorous way. This kind of statements is rather rare in the context of approximate solution of boundary value problems.

Modified versions of periodic successive approximations method convergent for arbitrarily large Lipschitz constants are given in [25] and [55], § 5.5. The idea there is to use other projection operators onto the space of periodic functions; in particular, constant functions do not belong to their kernels, in contrast to that used in (15) (due to the last property, one cannot obtain in this way the corollaries on the periods of periodic solutions of autonomous systems established in [43, 45, 48]). In [25], the Lipschitz condition for the non-linearity is assumed globally (i. e., for $G = \mathbb{R}^n$); this assumption is rather restrictive.

The Cesari method ([8, 9]; see also [67]) likewise provides one a way to reduce the periodic problem (11), (12) to a finite-dimensional system of equations. This method is based, in the notation of [24], on the use of the operator

$$H_m := \Lambda - P_m \Lambda$$

in a suitable space of p-periodic functions, where Λ is given by (73) with $a = 0$, $b = p$, m is fixed, and P_m computes the mth partial sum of the Fourier series of the corresponding function. The number of resulting determining equations depends on the Lipschitz

[17] Shooting schemes mostly deal with two-point boundary conditions; see, e.g., [3, 5, 11, 15, 17, 22, 42, 75]. Additional assumptions are often required; e.g., in [17] the boundary condition $Au(a) + Bu(b) = c$ can be treated on the assumption that $\operatorname{rank} A + \operatorname{rank} B = n$.

[18] Note that, in our approach, the numerical solution of approximate determining equations is considered as a separate problem and any smooth techniques, whenever applied for this purpose, are not a necessary part of the method.

constant of f so that it grows with m, and the convergence is guaranteed for m large enough. The type of convergence is discussed in [24].

One may note that the application of Cesari's approach in concrete situations is not an easy task. The choice of m is, generally speaking, not constructive (it is shown that such a sufficiently large m exists; in [24, Lemma 2], where the convergence in the C-norm is proved, no bound is specified for the value $Q(\alpha, T)$). As regards Hale's setting [16], a different meaning of parameters implies that the initial value of a theoretically detected periodic solution remains unknown (in fact, one claims the existence of a periodic solution with a particular integral mean value, while its initial value is still to be determined; more details can be found in [65], § A2.2). In the approach presented here, apart from the integral mean value, one does not need to compute any higher order terms in the Fourier expansion. Furthermore, the technique is rather easily adopted for other problems. The interval division appears to be a simpler and computationally less demanding way to guarantee the convergence which, in addition, may simultaneously serve other purposes.

§ 6. Extensions and other problems. The techniques based on § § 1-4, after suitable modifications, are applicable to problems other than the periodic one.

In particular, the approach is rather easy to be adopted for more general two-point boundary value problems [61, Theorem 12]; for this purpose, it is sufficient to modify only the form of the starting approximation and take care of a suitable choice of the admissible domain. All the essential properties of the iterations are retained and, in particular, the given boundary condition is satisfied at every step. Solvability conditions can be obtained similarly to the periodic case (cf. [57]). If we relax the requirements by admitting that the last mentioned property can be satisfied only approximately, we can construct a scheme for the study of a general non-local boundary value problem as in [63].

Ideas from [49, 58, 61] can be efficiently used, in particular, in the study of solutions of differential equations possessing a given number of zeroes. In [56], we describe the approach for the Dirichlet problem for a second-order equation focusing on solutions u with type $(\sigma_0, \sigma_1; t_1)$ (i. e., such that $u(t_1) = 0$, $\operatorname{sign} u = \sigma_0$ on $[a, t_1]$, and $\operatorname{sign} u = \sigma_1$ on $[t_1, b]$; see Definition IV.2), where the values of signs σ_0, σ_1 are fixed but the value of t_1 is unknown and should be determined together with the solution u. A more general situation is treated in Chapter IV.

Further interesting applications are related to systems with impulses at variable times, in which jumps occur when the trajectory meets a certain surface. In such cases, the approximate integration of the equation is rather difficult even under initial conditions only. It turns out that parametrisation techniques work quite well for this kind of problems. This topic is discussed in Chapter III.

The development of similar techiques for functional differential equations is work in progress; some results in this direction are obtained in [50, 52, 57, 64].

II. Parametrisation for a non-local problem.

The chapter is based on [58, 62, 63].

Following [62, 63], consider the boundary value problem

$$u'(t) = f(t, u(t)), \qquad t \in [a, b], \tag{66}$$

$$\phi(u) = \gamma, \tag{67}$$

where $\phi : C([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is a vector functional, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a function satisfying the Carathéodory conditions in a certain bounded set, and γ is a given vector. By a solution of the problem, one means an absolutely continuous function with property (67) satisfying (66) almost everywhere on $[a, b]$.

In [63], we adjust the parametrisation scheme of the kind described in Chapter I for use with general non-local boundary conditions (67). The idea is based on the reduction of (66), (67) to a family of simpler auxiliary problems with two-point linear separated conditions at a and b:

$$u(a) = \xi, \tag{68}$$

$$u(b) = \eta, \tag{69}$$

where ξ and η are unknown parameters. By doing so one can use, in the non-local case, the techniques adopted to two-point problems. Further on, in [62], we describe the version using the polynomial interpolation which is better suited for practical computations (§ 10).

Here, in contrast to Chapter I, the fulfilment of the boundary condition for the iterations is not guaranteed any more: it is satisfied only approximately for the approximate solutions obtained at every step. On the other hand, the process can be applied directly to a wider range of boundary conditions; their particular properties are essential in the analysis of determining equations. Other related approaches to boundary value problems with various non-linear conditions are discussed in [55].

§ 7. Parametrisation using auxiliary two-point problems. Let us fix certain bounded sets $G_i \subset \mathbb{R}^n$, $i = 0, 1$, and focus on the solutions u of problem (66), (67) with $u(a) \in G_0$ and $u(b) \in G_1$.

Definition II.1. For any two sets G_0 and G_1 in $\mathbb{R}^n$, we put

$$\mathscr{C}(G_0, G_1) := \{\theta\xi + (1 - \theta)\eta : \xi \in G_0, \eta \in G_1, \theta \in [0, 1]\}.$$

The working domain for our problem is constructed as follows: for the given G_0 and G_1, construct the set

$$G := \mathscr{C}(G_0, G_1) \tag{70}$$

and put $G_\varrho := \mathscr{O}_\varrho(G)$ for $\varrho \in \mathbb{R}^n_+$ (see (6)). Set (70) will serve as the domain for admissible values of parameters. By analogy with § I, define the parametrised iterations

$\{u_m(\cdot,\xi,\eta) : m \geq 0\}$ by putting

$$u_0\,(t,\xi,\eta) := \left(1-\frac{t-a}{b-a}\right)\xi + \frac{t-a}{b-a}\eta, \tag{71}$$

$$u_m\,(t,\xi,\eta) = u_0(t,\xi,\eta) + (\Lambda N_f u_{m-1}(\cdot,\xi,\eta)])(t), \tag{72}$$

for $t \in [a,b]$, $m = 1,2,\ldots$. In (72), Λ is the linear operator in $C([a,b],\mathbb{R}^n)$ given by the formula

$$(\Lambda y)\,(t) := \int_a^t y(s)ds - \frac{t-a}{b-a}\int_a^b y(s)ds, \qquad t \in [a,b], \tag{73}$$

and N_f is the Nemytskii operator generated by the non-linearity from (66):

$$(N_f y)\,(t) := f(t,y(t)), \qquad t \in [a,b], \tag{74}$$

for any continuous $y : [a,b] \to \mathbb{R}^n$.

Theorem II.2. *Let there exist a non-negative vector ϱ satisfying the inequality*

$$\varrho \geq \frac{b-a}{4}\delta_{G_\varrho}(f), \tag{75}$$

such that $f \in \mathrm{Lip}_K(G_\varrho)$ with a matrix K for which

$$r(K) < \frac{1}{\gamma_0(b-a)}, \tag{76}$$

where $\gamma_0 = 3/10$. Then, for all fixed $(\xi,\eta) \in G_0 \times G_1$:

1. *The limit $\lim_{m\to\infty} u_m(t,\xi,\eta) =: u_\infty(t,\xi,\eta)$ exists uniformly in $t \in [a,b]$, and $u_\infty(\cdot,\xi,\eta)$ is the unique solution of problem* (66), (68), (69).
2. *$u_\infty(t,\xi,\eta) \in G_\varrho$ and the estimate*

$$|u_\infty(t,\xi,\eta) - u_m(t,\xi,\eta)| \leq \frac{5}{9}\alpha_1(t)Q^m\left(\mathbf{1}_n - Q\right)^{-1}\delta_{G_\varrho}(f) \tag{77}$$

holds for any $t \in [a,b]$ and $m \geq 0$,[19] where

$$Q := \gamma_0(b-a)K. \tag{79}$$

Under conditions (75) and (76), one can characterise the solvability of the two-point problem with separated conditions (69) in terms of function $u_\infty(\cdot,\xi,\eta)$. Consider the system with constant forcing term

$$u'(t) = f(t,u(t)) + \mu(b-a)^{-1}, \qquad t \in [a,b], \tag{80}$$

where $\mu = \mathrm{col}\,(\mu_1,\ldots,\mu_n)$. Define $\Delta : G_0 \times G_1 \to \mathbb{R}^n$ by putting

$$\Delta(\xi,\eta) := \eta - \xi - \int_a^b f(s,u_\infty(s,\xi,\eta))ds \tag{81}$$

for $(\xi,\eta) \in G_0 \times G_1$.

[19] In (77), $\alpha_1(\cdot) = \alpha_1(\cdot,a,b)$ (see (9), p. 2). Clearly,

$$\alpha_1(t) = 2(t-a)\left(1-\frac{t-a}{b-a}\right), \qquad t \in [a,b]. \tag{78}$$

Theorem II.3. *Let $\xi \in G_0$ and $\eta \in G_1$ be fixed. Let there exist a non-negative vector ϱ wit property* (75) *such that $f \in \mathrm{Lip}_K(G_\varrho)$ with a matrix K satisfying* (76). *Then for a solution of the initial value problem* (80), (68) *to have property* (69) *it is neccessary and sufficient that*

$$\mu = \Delta(\xi, \eta). \tag{82}$$

Moreover, in the case where (82) *holds, the solution of problem* (80), (68) *coincides with $u_\infty(\cdot, \xi, \eta)$.*

It follows from Theorem II.3 that solutions of the non-local boundary value problem (66), (67) have form $u_\infty(\cdot, \xi, \eta)$, where ξ and η are determined from the equations

$$\Delta(\xi, \eta) = 0, \tag{83}$$

$$\phi(u_\infty(\cdot, \xi, \eta)) = d. \tag{84}$$

Theorem II.4. *Let there exist a non-negative vector ϱ with property* (75) *such that $f \in \mathrm{Lip}_K(G_\varrho)$ with a matrix K for which* (76) *holds.*

1. *If there exists a pair $(\xi, \eta) \in G_0 \times G_1$ satisfying* (83) *and* (84), *then the boundary value problem* (66), (67) *has a solution $u(\cdot)$ such that* (68), (69) *hold and*

$$\{u(t) : t \in [a, b]\} \subset G_\varrho. \tag{85}$$

2. *If the boundary value problem* (66), (67) *has a solution $u(\cdot)$ such that* (85) *holds, then the pair $(u(a), u(b))$ is a solution of system* (83), (84).

The last statement, which resembles Theorem I.3 for the periodic case, reduces the problem to the study of determining equations (83), (84).

The practical analysis of problem (66), (67) is carried out by analogy with § 2 using approximate versions of the determining system (83), (84):

$$\Delta_m(\xi, \eta) = 0, \tag{86}$$

$$\phi(u_m(\cdot, \xi, \eta)) = d, \tag{87}$$

where m is fixed and $\Delta_m : G_0 \times G_1 \to \mathbb{R}^n$ is given by the relation

$$\Delta_m(\xi, \eta) := \eta - \xi - \int_a^b f(s, u_m(s, \xi, \eta))ds \tag{88}$$

for all $(\xi, \eta) \in G_0 \times G_1$. Approximations to solutions of (66), (67) possessing properties (68), (69) and (85) are constructed in the form

$$U_{m_0}(t) := u_{m_0}(t, \tilde{\xi}, \tilde{\eta}), \qquad t \in [a, b], \tag{89}$$

where m_0 is fixed and $(\tilde{\xi}, \tilde{\eta})$ is a root of (86), (87) with $m = m_0$. Solvability conditions involving the mappings $(\xi, \eta) \mapsto (\Delta_m(\xi, \eta), \phi(u_m(\cdot, \xi, \eta))$, under additional assumptions on ϕ, can be formulated by analogy to [61].

§ 8. The case of large Lipschitz constants. If condition (76), § 7, is violated, then the argument from § 4 can be applied. In this case, we choose k_0 so that the inequality

$$r(K) < \frac{2^{k_0}}{\gamma_0(b - a)} \tag{90}$$

holds and, increasing k_0 if needed, determine ϱ from the condition

$$\varrho \geq \frac{b - a}{2^{k_0 + 2}} \, \delta_{G_\varrho}(f). \tag{91}$$

As a result, at the expense of an increased number of variables in equations, we obtain a convergent scheme with similar properties.

§ 9. Freezing at multiple nodes. Interval divisions need not be carried out at the ratio $1 : 2^{k_0}$ (§ 4). The general position of nodes, although leading to somewhat cumbersome formulae, may however be useful in certain situations (in particular, if a change of behaviour at the corresponding nodes is expected; two problems of this kind are discussed in Chapters III and IV).

The corresponding formulae are derived [58] by analogy with § 4. In this case, we choose the nodes

$$t_0 = a, \qquad t_k = t_{k-1} + h_k, \qquad k = 1, \dots, N-1, \qquad t_N = b, \tag{92}$$

where $N > 1$ and $h_k > 0$, $k = 1, \dots, N-1$, and "freeze" the values of u at points (92) by formally putting

$$u(t_k) = z^{(k)}, \qquad k = 0, 1, \dots, N. \tag{93}$$

Then we consider the restrictions of equation (66) to the subintervals

$$u'(t) = f(t, u(t)), \qquad t \in [t_{k-1}, t_k]\,, \tag{94}$$

where $k = 1, 2, \dots, N$, under the natural two-point boundary conditions[20]

$$u(t_{k-1}) = z^{(k-1)}, \quad u\,(t_k) = z^{(k)}. \tag{95}$$

The non-local problem (66), (67) is then reformulated as determining the parameters $z^{(0)}$, $z^{(1)}$, ..., $z^{(N)}$ so that (67) holds, where $u : [a, b] \to \mathbb{R}^n$ is the result of a continuous gluing of solutions of problems (94), (95) on the subintervals of the division.

Before constructing iterations, similarly to § 7, we select some admissible domains for "frozen" values by choosing closed bounded sets $G_k \subset \mathbb{R}^n$, $k = 0, 1, \dots, N$, and focus on solutions of two-point problems (94), (95) such that

$$u(t_k) \in G_k, \qquad k = 0, 1, \dots, N. \tag{96}$$

Similarly to (70), we form the sets

$$G_{k-1,k} := \mathscr{C}(G_{k-1}, G_k), \qquad k = 1, 2, \dots, N, \tag{97}$$

and, for any non-negative vector ϱ, put $G_k(\varrho) := \mathscr{O}_\varrho(G_{k-1,k})$, $k = 1, 2, \dots, N$.

The formulae for iterations are constructed similarly to §§ 1, 7: for any fixed $z^{(0)}$, $z^{(1)}, \dots, z^{(N)}$, we define $u_m^{(k)} : [t_{k-1}, t_k] \times G_{k-1} \times G_k \to \mathbb{R}^n$, $k = 1, 2, \dots, N$, $m = 0, 1, 2, \dots$, by putting

$$\begin{aligned} u_m^{(k)}\big(t, z^{(k-1)}, z^{(k)}\big) = u_0^{(k)}\big(t, z^{(k-1)}, z^{(k)}\big) + \int_{t_{k-1}}^{t} f\Big(s, u_{m-1}^{(k)}\big(s, z^{(k-1)}, z^{(k)}\big)\Big)\, ds \\ - \frac{t - t_{k-1}}{h_k} \int_{t_{k-1}}^{t_k} f\Big(s, u_{m-1}^{(k)}\big(s, z^{(k-1)}, z^{(k)}\big)\Big)\, ds \end{aligned} \tag{98}$$

for all $m = 1, 2, \dots$ and $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N$, where the graphs of the functions $u_0^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}\big)$, $k = 1, 2, \dots, N$, form a broken line joining the points $(t_k, z^{(k)})$, $k = 1, 2, \dots, N$ (cf. note 13, p. 12). We then prove the uniform convergence of iterations (98) (Theorem 5.1, p. 111) and formulate a reduction principle by analogy with § 1 (Theorem 6.1, p. 115).

The techniques are applicable under the following assumptions:

[20] One may notice that, for any fixed k, the two-point problem (94), (95) is, strictly speaking, overdetermined. It should be noted, however, that (94), (95) are considered simultaneously for all $k = 1, 2, \dots, N$, with the terminal condition serving as an initial condition on the next interval.

1. There exist non-negative vectors $\varrho^{(1)}$, $\varrho^{(2)}$, …, $\varrho^{(N)}$ such that

$$\varrho^{(k)} \geq \frac{h_k}{4} \delta_{[t_{k-1}, t_k], G_k(\varrho^{(k)})}(f) \tag{99}$$

for all $k = 1, 2, \ldots, N$ (see notation (3)).

2. There exist non-negative matrices K_1, K_2, …, K_N such that

$$f \in \mathrm{Lip}_{K_k}\big(G_k(\varrho^{(k)})\big), \qquad k = 1, 2, \ldots, N. \tag{100}$$

3. The spectral radii of K_1, K_2, …, K_N satisfy the inequality

$$r(K_k) < \frac{1}{\gamma_0 h_k}, \qquad k = 1, 2, \ldots, N. \tag{101}$$

This approach, in which the solution is approximated simultaneously on multiple intervals, may be particularly useful in the cases where the related numerical methods (namely, parallel shooting [30, 75]) cannot be applied (see § 5).

§ 10. Polynomial interpolation. The main difficuly in the application of the techniques from § 7 is the analytic integration of expressions involving variable parameters. In [62], we formulate and justify the polynomial version of the method from [63]. The resuting scheme, which is significantly easier for practical realisation, can be used in the cases where the approach of § 7 faces with computational difficulties.

Consider the non-local problem (66), (67), where $\phi : C([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is a non-linear vector functional and $\gamma \in \mathbb{R}^n$ is a given vector. In contrast to § 7, we assume that $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in a certain bounded set. We also need an additional regularity assumption on f with respect to the time variable.

By a solution of problem (66), (67) we understand here a continuously differentiable vector function with property (67) satisfying (66) everywhere on $[a, b]$. This is motivated by the additional Hölder continuity assumption in the time variable imposed below.

In order to proceed, introduce some notation. Fix a natural number q and set

$$L_q y := \mathrm{col}(L_q y_1, L_q y_2, \ldots, L_q y_n) \tag{102}$$

for any continuous $y : [a, b] \to \mathbb{R}^n$, where $L_q y_i$ is the qth degree interpolation polynomial for y_i at the Chebyshev nodes

$$t_i = \frac{b-a}{2} \cos \frac{(2i-1)\,\pi}{2\,(q+1)} + \frac{a+b}{2}, \qquad i = 1, 2, \ldots, q+1, \tag{103}$$

translated from $(-1, 1)$ to the interval (a, b) (see, e. g., [41]).

For $y \in C([a, b], \mathbb{R}^n)$, put

$$E_q y = \mathrm{col}(E_q y_1, E_q y_2, \ldots, E_q y_n),$$

where $E_q(x)$ is the error of the best uniform approximation of a function x by polynomials of degree $\leq q$: $E_q(y) := \inf_{p \in \mathscr{P}_q} \max_{t \in [a,b]} |y(t) - p(t)|$, where $\mathscr{P}_q$ the set of all polynomials of degree not higher than q on $[a, b]$ [13, 40].

If $D \subset \mathbb{R}^n$ is a closed domain and $f : [a, b] \times D \to \mathbb{R}^n$, put

$$l_{q,D}(f) := \left(\frac{2}{\pi} \ln q + 1\right) \sup_{p \in \mathscr{P}_{q+1,D}} E_q(N_f p), \tag{104}$$

where

$$\mathscr{P}_{q,D} := \{u : u \in \mathscr{P}_q^n,\ u([a,b]) \subset D\}$$

with $\mathscr{P}_q^n := \mathscr{P}_q \times \cdots \times \mathscr{P}_q$.[21] Introduce a modified iteration process keeping formula (71) for the starting approximation:

$$v_0^q(\cdot,\xi,\eta) := u_0(\cdot,\xi,\eta) \tag{105}$$

and replacing (72) by the formula

$$v_m^q(t,\xi,\eta) := u_0(t,\xi,\eta) + (\Lambda L_q N_f v_{m-1}^q(\cdot,\xi,\eta))(t) \tag{106}$$

for $t \in [a,b]$, $m = 1, 2, \ldots$.[22]

Theorem II.5. *Let there exist a non-negative vector* ϱ *such that*

$$\varrho \geq \frac{b-a}{4}\left(\delta_{G_\varrho}(f) + 2l_{q,G_\varrho}(f)\right) \tag{107}$$

and (32) *holds with a certain matrix* K *satisfying* (76). *Furthermore, let there exist vectors* c *and* β *with* $c_i \geq 0$, $0 < \beta_i \leq 1$, $i = 1, 2, \ldots, n$, *such that*[23]

$$f(\cdot,\xi) \in H_c^\beta \tag{108}$$

for all fixed $\xi \in G_\varrho$. *Then, for all fixed* $(\xi,\eta) \in G_0 \times G_1$ *:*

1. For any $m \geq 0$, $q \geq 1$, *the function* $v_m^q(\cdot,\xi,\eta)$ *is a vector polynomial of degree* $q+1$ *having values in* G_ϱ *and satisfying the two-point conditions* (69).

2. The limits

$$v_\infty^q(\cdot,\xi,\eta) := \lim_{m\to\infty} v_m^q(\cdot,\xi,\eta), \qquad v_\infty(\cdot,\xi,\eta) := \lim_{q\to\infty} v_\infty^q(\cdot,\xi,\eta) \tag{109}$$

exist uniformly on $[a,b]$. *Functions* (109) *satisfy conditions* (69).

3. The estimate

$$|u_\infty(t,\xi,\eta) - v_m^q(\cdot,\xi,\eta)| \leq \frac{10}{9}\alpha_1(t) Q^m (\mathbf{1}_n - Q)^{-1}\left(\delta_{G_\varrho}(f) + l_{q,G_\varrho}(f)\right)$$

holds for any $t \in [a,b]$, $m \geq 0$, *where* Q *and* α_1 *are given by* (79), (78).

The inconvenience of (107), which is due to the difficulties in estimating numbers (104), can be compensated by adjusting the degree of the polynomial. Put

$$w_0(\cdot,\xi,\eta) := u_0(\cdot,\xi,\eta), \tag{110}$$

$$w_m(t,\xi,\eta) := u_0(t,\xi,\eta) + (\Lambda L_{q_m} N_f w_{m-1}(\cdot,\xi,\eta))(t) \tag{111}$$

for $t \in [a,b]$, $m = 1, 2, \ldots$, where $\{q_m : m \geq 1\} \subset \mathbb{N}$. Replace (107) by the following condition: there exist a non-negative vector ϱ and a strictly positive r such that

$$\varrho \geq \frac{b-a}{4}\left(\delta_{G_\varrho}(f) + r\right). \tag{112}$$

[21] The second multiplier in (104) is the least upper bound of errors of best uniform approximations of the functions obtained by substitution into the right-hand side of equation (66) of vector polynomials of degree $\leq q+1$ with values in D.

[22] For any $q \geq 1$, formula (106) defines a vector polynomial $v_m^q(\cdot,\xi,\eta)$ of degree $\leq q+1$ (in particular, all these functions are continuously differentiable), which, moreover, satisfies the two-point boundary conditions (69). The coefficients of the polynomials depend on the parameters ξ and η.

[23] See notation 14, p. 2.

Theorem II.6. *Assume that f satisfies the Lipschitz condition* (32) *on G_ϱ, where ϱ is such that* (112) *holds with some r. If, moreover, condition* (76) *is satisfied, then sequence* (110), (111) *uniformly converges on $[a, b]$ provided that q_m is chosen large enough at every step m.*

The last theorem implies that, for suitably chosen $\{q_m : m \geq 1\}$, sequence (110), (111) serves the same purpose as sequence (105), (106) under the assumptions of Theorem II.5.

Theorem II.6 does not provide precise information about how large the degree of polynomials should be at individual steps. This formulation can nevertheless be sufficient for the practical application because one can start checking the behaviour of approximations heuristically with relatively small degrees of polynomials, expecting still better results for the explicitly unknown "guaranteed" degrees q_m.[24]

On the other hand, a question arises on the possible behaviour of the sequence $\{q_m : m \geq 1\}$ for large m and especially on the finiteness of $\sup_{m\geq 1} q_m$. It turns out that under a smallness condition somewhat stronger than (76), the sequence of degrees of polynomials appearing in (111) can always be chosen bounded. More precisely, suppose that, instead of (76), the matrix K appearing in (32) satisfies the condition

$$r(K) < \frac{2}{b-a} \tag{113}$$

(i. e., the constant 10/3 is replaced by 2). We assume that $\beta_i = 1$, $i = 1, 2, \ldots, n$, in (108).

The last theorem of [62] states that, under condition (113), one can choose the same degree of polynomial in (111) at every step.

Theorem II.7. *Let there exist a non-negative vector ϱ and positive vector r such that* (112) *holds and the Lipschitz condition* (32) *holds with K satisfying* (113). *Assume that $f(\cdot, \xi)$ is Lipschitzian with some constant vector c for all fixed $\xi \in G_\varrho$. Then the iteration process* (110), (111) *uniformly converges on $[a, b]$ if $q_m = q$, $m = 1, 2, \ldots$, with q sufficiently large.*

In other words, under conditions of Theorem II.7, the iteration process (110), (111) reduces to (105), (106) with q large enough. In the cases where (113) is violated, one can apply interval divisions by analogy with § 4 (see § 8).[25]

The key feature here is the ease of application in obtaining higher-order iterations, which allows us to get high quality approximations with much less computational efforts. Polynomial approximations prove to be particularly helpful for differential equations with argument deviations where, due to the character of the equation, the computations are more complicated [64].

[24] For practical computations, the use of interpolation polynomials of very large degrees on the entire interval is known to be unsatisfactory due to the accumulation of rounding errors. The use of interval divisions is rather natural in such cases, which corresponds to piecewise polynomial interpolation ([68], Chapter 3, § 4).

[25] Strictly speaking, under conditions of Theorem II.5, estimates (90), (91) should be modified accordingly due to condition (113). However, similarly to § 4, the resulting conditions are also satisfied for sufficiently large number of intermediate nodes.

III. Approximations for problems with state-dependent jumps.

The chapter is based on [36, 49].

§ 11. Overview. The parametrisation techniques based on the ideas expressed above are rather flexible and can be adopted to many problems of various nature. In [36, 49], we show how they can be used to study boundary value problems for impulsive systems where jumps may occur at times depending on the value of the solution itself (i. e., the so-called state-dependent jumps; we refer to the book [37] for details on this relatively new and little studied subject). We suggest a reduction technique of Lyapunov-Schmidt type and describe the construction of approximate solutions. Existence conditions are not treated here (although their obtaining is possible by analogy to [52, 61]).

Differential models involving state-dependent jumps are of much interest since they arise in a number of applications (see, e. g., [10, 18, 34, 77, 79]). The majority of currently available results on impulsive boundary value problems concern systems where impulses occur at fixed times.

At present, according to our knowledge, no constructive approaches for boundary value problems with state-dependent impulses are available in the literature. We can mention only the work [21], where the author suggests MATHEMATICA packages for the practical analysis of certain particular classes of two and three-dimensional systems with variable jumps, and the recent paper [3] describing the simple shooting procedure for a state-dependent two-point problem with one impulse position. The approach of these works is numerical.

It turns out that there is a relatively simple way to treat this topic using ideas from [58, 61] (§§ 4, 9) which, due to the nature of the impulse action at variable times, are particularly natural here.

The existing literature concerning boundary value problems for systems with state-dependent jumps is mostly devoted to the study of initial value problems and the periodic problem. For other types of boundary conditions, one cannot name but a few related works since the majority of results on boundary value problems for impulsive systems concern jumps at fixed times. We refer to [36, 49] for the related bibliography.

One may note that, for equations with state-dependent jumps, boundary value problems have been studied mostly in the case where the trajectory is allowed to meet the barrier surface only once (see, e. g., [37], § 6.1, or [3]). In [49], we show that the situation with multiple intersections with the barrier can also be treated by using a suitable parametrisation technique (§ 12).

§ 12. Barrier, jump condition, and definition of solution. In [36, 49], we are interested in the approximate construction of solutions of the non-linear system of differential equations

$$u'(t) = f(t, u(t)), \qquad t \in [a, b], \tag{114}$$

with $-\infty < a < b < \infty$ and a continuous $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, satisfying a two-point boundary condition and a jump condition with non-fixed jump times. In [49], we consider system (114) under the non-linear two-point boundary condition

$$h(u(a), u(b)) = 0, \tag{115}$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, and impose the jump condition

$$u(t+) - u(t-) = \gamma_t (u (t-)) \ \text{ for } t \in (a, b) \text{ such that } g (t, u (t-)) = 0. \tag{116}$$

The functions $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in (a, b)$, appearing in (116) are also assumed to be continuous. In the context of systems (114) with impulse action at variable times, the set

$$\mathscr{B} = \{(t, x) \in [a, b] \times \mathbb{R}^n : g(t, x) = 0\} \tag{117}$$

determined by the function g from (116) is usually called a barrier; it contains all the points of the phase space where the jumps occur. The time instants $t \in (a, b)$ for which (116) applies are *a priori* unknown; they are referred to as state-dependent because they depend on the solution u itself through the equation $g(t, u(t-)) = 0$.

In [49], we study solutions of problem (114), (115), (116) that are allowed to meet the barrier p times with $1 \leq p < \infty$.[26]

Definition III.1. Let $p \in \mathbb{N}$. A left continuous vector function $u : [a, b] \to \mathbb{R}^n$ is called a *solution* of problem (114), (115), (116) *with p jumps* if (115) holds and there exist points $a < \tau_1 < \tau_2 < \cdots < \tau_p < b$ such that the restrictions $u|_{[a,\tau_1]}$, $u|_{(\tau_1,\tau_2]}$, ..., $u|_{(\tau_p,b]}$ have continuous derivatives, u satisfies (114) and the relation $g(t, u(t)) \neq 0$ for $t \in [a, b] \setminus \{\tau_1, \tau_2, \ldots, \tau_p\}$, and

$$g(\tau_i, u(\tau_i)) = 0, \quad u(\tau_i+) - u(\tau_i) = \gamma_{\tau_i}(u (\tau_i)), \quad i = 1, \ldots, p. \tag{118}$$

We see that, for any $i = 1, \ldots, p$, the trajectory of u meets the barrier $\mathscr{B}$ at the time τ_i (i.e., $(\tau_i, u(\tau_i)) \in \mathscr{B}$) and undergoes a jump of size $\gamma_{\tau_i}(u(\tau_i))$. The time instants $\tau_1, \ldots, \tau_p$, as well as their number p, both depend on u, so that different solutions may have jumps at different points.

Our approach uses ideas from [58, 63] (§§ 7, 9) and is based on the construction of simple model problems involving unknown parameters. Under certain conditions one shows that, for all values of parameters from suitable bounded sets, solutions of the auxiliary problems can be obtained as limits of uniformly convergent successive approximations. The values of parameters should then be found from the corresponding determining equations generated by the functional perturbation terms, boundary conditions, and the barrier crossing condition. As well as in the previously discussed cases, the practical analysis is based on the study of approximate determining equations, which are solved numerically in the selected regions. Approximate solutions of (114)–(116) are then constructed from those of the model problems. The computations for this kind of problems are more tedious and it is natural to carry out them by using computer algebra systems (e. g., MAPLE 14, which has been applied in [36]).

According to our knowledge, the scheme suggested in [36, 49] is the first numerical-analytic method for this type of impulsive problems. It can be applied for problems with either linear or non-linear boundary conditions.[27] Furthermore, in this way, we can consider barriers described in the implicit form (117), in contrast to the most frequently studied case where it is given by an equation expicitly resolved with respect to the time variable, i. e., has the form $\mathscr{B} = \{(t, x) : t = \tilde{g}(x)\}$ (see, e. g., [38, 39]). The method also allows one to treat multiple solutions of the problem.

[26] In particular, the pulse phenomenon [4] is excluded from consideration.

[27] For problems without impulses, this is shown in [50, 53, 55, 61, 63, 65, 66].

§ 13. Auxiliary problems and construction of iterations. Let $p \geq 1$ be fixed. A solution u of problem (114)–(116) with p jumps is approximated by suitable iteration sequences separately on the interval $[a, \tau_1]$, which precedes the jump times $\tau_1, \tau_2, \ldots, \tau_p$ (pre-jump evolution) and then sequentially on the intervals $[\tau_1, \tau_2]$, $[\tau_2, \tau_3]$, …, $[\tau_{p-1}, \tau_p]$, with $\tau_{p+1} := b$, which correspond to the after-jump evolution. The time instants where the jumps occur, $\tau_1, \tau_2, \ldots, \tau_p$, and the values $\lambda^{[1]}$, $\lambda^{[2]}$, …, $\lambda^{[p]}$ are treated as parameters. The latter have the meaning of pre-jump values of the solution, whereas ξ and $\lambda^{[p+1]}$ stand for the values of the solution at a and b.

The auxiliary problems are constructed as follows. Choose $p + 2$ compact sets G_0, G_1, …, G_p, G_{p+1} in $\mathbb{R}^n$ and, applying a shift by γ_{τ_k} from (116), define the sets

$$G_k^+ := \{x + \gamma_{\tau_k}(x) : x \in G_k\}, \quad k = 1, \ldots, p.$$

These sets will determine an admissible space localisation of the trajectory at the corresponding times, namely, we focus on solutions u of problem (114), (115), (116) with p jumps such that

$$u(a) \in G_0; \quad u(\tau_k) \in G_k, \quad u(\tau_k+) \in G_k^+, \quad k = 1, \ldots, p; \quad u(\tau_{p+1}) \in G_{p+1}.$$

Similarly to § 1, we need to impose assumptions on sets somewhat wider than the admissible regions. To define suitable neighbourhoods of sets where the values of parameters are looked for, choose some vectors $\varrho^{[i]}$, $i = 0, 1, \ldots, p$ from $\mathbb{R}^n_+$ and, using notation (6), p. 1, construct the sets

$$V_0(\varrho^{(0)}) := \mathscr{O}_{\varrho^{(0)}}(\mathscr{C}(G_0, G_1)), \quad V_k(\varrho^{(k)}) := \mathscr{O}_{\varrho^{(k)}}(\mathscr{C}(G_k^+, G_{k+1})), \quad k = 1, \ldots, p. \tag{119}$$

Then we introduce the auxiliary two-point boundary value problems

$$\begin{gathered} x'(t) = f(t, x(t)), \qquad t \in [a, \tau_1], \\ x(a) = \xi, \quad x(\tau_1) = \lambda^{[1]} \end{gathered} \tag{120}$$

and

$$\begin{gathered} y'(t) = f\big(t, y(t)\big), \qquad t \in [\tau_k, \tau_{k+1}], \\ y(\tau_k) = \lambda^{[k]} + \gamma_k\big(\lambda^{[k]}\big), \qquad y(\tau_{k+1}) = \lambda^{[k+1]} \end{gathered} \tag{121}$$

for $k = 1, 2, \ldots, p$. The variables

$$\begin{gathered} a < \tau_1 < \cdots < \tau_p < \tau_{p+1} = b, \quad \xi = \operatorname{col}(\xi_1, \xi_2, \ldots, \xi_n) \in G_0, \\ \lambda^{[k]} = \operatorname{col}\big(\lambda_1^{[k]}, \lambda_2^{[k]}, \ldots, \lambda_n^{[k]}\big) \in G_k, \qquad k = 1, 2, \ldots, p+1, \end{gathered} \tag{122}$$

appearing in (120) and (121) are considered as unknown parameters. We consider only those solutions $x(\cdot)$ and $y^{[k]}(\cdot)$, $k = 1, \ldots, p$, of problems (120) and (121) which have range in the sets $V_0(\varrho^{(0)})$ and $V_k(\varrho^{(k)})$ (see (119)).[28]

Appproximations are constructed separately on the ineryal $[a, \tau_1]$, where still no jumps occur, and each of the p intervals corresponding to the system evolution between jumps. Problem (120) on $[a, \tau_1] \times V_0(\varrho^{(0)})$ corresponds to the pre-jump evolution, and we treat it by using the parameterised sequences of vector functions

$$x_m(t) := x_m\big(t; \tau_1, \xi, \lambda^{[1]}\big), \quad t \in [a, \tau_1], \ m \geq 1, \tag{123}$$

[28] Formally speaking, (120) and (121) are both overdetermined problems (with n equations and $2n$ boundary conditions). However, similarly to § 9 (see note 20, p. 19), one can see that this does not cause any complications since (120) and (121) are treated simultaneiusly.

with the parameters $\tau_1, \xi, \lambda^{[1]}$ from (122), by the relations

$$x_0(t) = \xi + \frac{t-a}{\tau_1 - a}\big(\lambda^{[1]} - \xi\big), \tag{124}$$

$$x_m(t) = x_0(t) + \int_a^t f(s, x_{m-1}(s))ds - \frac{t-a}{\tau_1 - a}\int_a^{\tau_1} f(s, x_{m-1}(s))ds, \quad t \in [a, \tau_1], \tag{125}$$

for $m \geq 1$. A similar construction is then used for the after-jump times, i. e., after the first intersection with the barrier; the parameterised iterations corresponding to problem (121) on $[\tau_k, \tau_{k+1}] \times V_k(\varrho^{(k)})$, $1 \leq k \leq p$,

$$y_m^{[k]}(t) := y_m^{[k]}\big(t; \tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}\big), \quad t \in [\tau_k, \tau_{k+1}], \tag{126}$$

are constructed in the form

$$y_0^{[k]}(t) = \lambda^{[k]} + \gamma_{\tau_k}(\lambda^{[k]}) + \frac{t-\tau_k}{\tau_{k+1} - \tau_k}\big(\lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]})\big), \tag{127}$$

$$y_m^{[k]}(t) = y_0^{[k]}(t) + \int_{\tau_k}^t f\big(s, y_{m-1}^{[k]}(s)\big)ds - \frac{t-\tau_k}{\tau_{k+1} - \tau_k}\int_{\tau_k}^{\tau_{k+1}} f\big(s, y_{m-1}^{[k]}(s)\big)ds \tag{128}$$

for $m \geq 1$. The convergence of all the iterations is established ([49], Theorems 4.2, 4.4) by analogy to § 9 under the conditions

$$f \in \operatorname{Lip}_{K_i}(V_i(\varrho^{(i)})), \quad i = 0, \ldots, p, \tag{129}$$

where the vectors $\varrho^{(i)}$, $i = 0, \ldots, p$, are chosen according to the inequality[29]

$$\varrho^{(k)} \geq \frac{b-a}{4}\,\delta_f(V_k(\varrho^{(k)})), \quad k = 0, 1, \ldots, p, \tag{130}$$

and the eigenvalues of the Lipschitz matrices satisfy the estimate

$$r(K_i) < \frac{10}{3(b-a)}, \quad i = 0, \ldots, p. \tag{131}$$

Conditions (129)–(131) allow us to construct the sequence of functions $\{u_m : m \geq 0\}$ defined on the entire $[a, b]$ by the formula

$$u_m(t) := \begin{cases} x_m(t) & \text{if } t \in [a, \tau_1], \\ y_m^{[k]}(t) & \text{if } t \in (\tau_k, \tau_{k+1}],\ k = 1, 2, \ldots, p, \end{cases}$$

and consider the corresponding limit as m tends to ∞,

$$u_\infty(t) := \begin{cases} x_\infty(t) & \text{if } t \in [a, \tau_1], \\ y_\infty^{[k]}(t) & \text{if } t \in (\tau_k, \tau_{k+1}],\ k = 1, 2, \ldots, p. \end{cases} \tag{132}$$

Then $u_\infty : [a, b] \to V_0(\varrho^{(0)}) \cup V_1(\varrho^{(1)}) \cup \cdots \cup V_p(\varrho^{(p)})$ is a vector function depending on the parameters $\tau_1, \ldots, \tau_p \in (a, b)$, $\xi \in G_0$, and $\lambda^{[k]} \in G_k$, $k = 1, \ldots, p+1$. For suitable values of these parameters ([49], Theorem 5.1), function (132) is a solution of the original boundary value problem (114), (115), (116) with p jumps; this reduces the problem to the determining equations

$$\begin{gathered} \Psi_0(\tau_1, \xi, \lambda^{[1]}) = 0, \quad h\big(\xi, \lambda^{[p+1]}\big) = 0, \\ \Psi_k(\tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}) = 0, \quad g\big(\tau_k, \lambda^{[k]}\big) = 0, \quad k = 1, \ldots, p, \end{gathered} \tag{133}$$

[29] $\delta_f(V_0(\varrho^{(0)}))$ and $V_0(\varrho^{(0)})$ are defined according to (4), (119).

in the sense that any root $(\tau_1^*, \tau_2^*, \dots, \tau_p^*, \xi^*, \lambda^{[1]*}, \lambda^{[2]*}, \dots, \lambda^{[p+1]*})$ of (133) determines a solution of (114)–(116) via the functions

$$
\begin{aligned}
x_\infty^*(t) &:= x_\infty\bigl(t; \tau_1^*, \xi^*, \lambda^{[1]*}\bigr), \quad t \in [a, \tau_1^*], \\
y_\infty^{[k]*}(t) &:= y_\infty^{[k]}\bigl(t; \tau_k^*, \tau_{k+1}^*, \lambda^{[k]*}, \lambda^{[k+1]*}\bigr), \quad t \in [\tau_k^*, \tau_{k+1}^*],\ k = 1, \dots, p-1.
\end{aligned} \tag{134}
$$

When conditions (130), (131) (i. e.,) are not fulfilled, one can suggest to adjust the interval division procedure (§ § 4, 8, 9) for this case (for problems without impulses, this technique is described in [60, 61]).

Since the form of system (133) is not explicitly known, the practical realisation of this scheme, as well as in other similar cases, is based on approximations of the determining system (133) obtained from (133) by replacing x_∞ and $y_\infty^{[k]}$ by the iterations x_m and $y_m^{[k]}$ from (125) and (128), respectively. Illustrations of the implementation of the scheme in concrete examples with different forms of barriers and numbers of jumps can be found in [49] (pp. 158–167 of this text). Practical computations according to this algorithm can be combined with the polynomial interpolation (see § 10), which considerably facilitates the analytic part of the scheme without noticeble loss in accuracy. If the problem has multiple solutions with the required properties, they can be detected by limiting the numerical analysis of the approximate determining systems to specific regions. An initial guess for the choice of sets is obtained from the zeroth approximation.

The single-jump case (i. e., if the solution is allowed to touch the barrier (117) only once in the given time interval) is studied specifically in [36], where we consider (114) under the jump condition

$$
u(t+) - u(t-) = \gamma(u(t-)) \quad \text{for } t \text{ such that } g(t, u(t-)) = 0 \tag{135}
$$

and the linear boundary condition

$$
Au(a) + Cu(b) = d, \tag{136}
$$

where d is a constant vector, and A, C are constant matrices satisfying the condition $\operatorname{rank}[A, C] = n$. The non-singularity of these matrices is not required. The functions $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ in (135) are assumed to be continuous. Here, the trajectory of u meets the barrier $\mathscr{B}$ at a unique time instant τ and has a jump of size $\gamma(u(\tau))$. Although further intersections with the barrier are not allowed, there may exists multiple solutions meeting $\mathscr{B}$ at different points; for their detection one should localise the analysis of the approximate determining equations to specific regions.

IV. Solutions with prescribed number of zeroes.

The chapter is based on [35].

The ideas from [36, 49, 58, 61] can also be used in other situations. In [35], based on the parametrisation techniques with interval divisions developed in [58, 60, 61, 63], we construct a scheme suitable for finding solutions of boundary value problems vanishing at certain points which are not specified *a priori*. We use an idea from [56], where a scalar second order equation is considered under the homogeneous Dirichlet conditions.

The question on finding solutions of non-linear differential equations possessing a prescribed number of zeroes inside the given interval is interesting from many points of view (see, e. g., [7, 28, 31, 32] and the references therein). This is a rather complicated problem and its investigation is generally based on considerations of purely qualitative character which usually do not provide any means to obtain approximations to the solution in question. Further difficulties arise when the equation is studied under non-linear boundary conditions.

This topic is of considerable interest, in particular, for Emden-Fowler type equations [80], the behaviour of solutions of which is rather complicated [23]. For such equations, the existence of solutions with a given number of zeroes is studied, in particular, in [2, 19, 32]; the Nehari theory is used in [14, 19, 20, 29, 33]. One may observe that the methods used to construct this kind of solutions usually strongly depend on the specific form of an equation (e. g., the approach of [14] is based on the use of the lemniscate functions, i. e., the solutions of specific Cauchy problems for a certain second order equation with a cubic non-linearity).

Paramerisation methods provide some kind of a general "machinery" for finding solutions with a given number of sign changes, which is applicable, in particular, in the cases mentioned above and may serve as a useful complement to other approaches. Following [35], consider systems of n first-order ordinary differential equations

$$u'(t) = f(t, [u(t)]_+, [u(t)]_-)\,, \qquad t \in [a, b]\,, \tag{137}$$

under a general two-point boundary condition

$$g(u(a), u(b)) = d. \tag{138}$$

Here, $[u]_\pm$ for any $u = \operatorname{col}(u_1, \dots, u_n)$ stands for the vector $\operatorname{col}([u_1]_\pm, \dots, [u_n]_\pm)$, and $[s]_+ := \max\{s, 0\}$, $[s]_- := \max\{-s, 0\}$ for any real s.

The functions $f\colon [a, b] \times G \times G \to \mathbb{R}^n$, $g : G \times G \to \mathbb{R}^n$ are assumed to be continuous.

Definition IV.1. Let $\{\sigma_0, \sigma_1\} \subset \{-1, 1\}$ and t_1 be a point from (a, b). We say that a function $u : [a, b] \to \mathbb{R}$ is of type $(\sigma_0, \sigma_1; t_1)$ if $u(t_1) = 0$ and

$$\sigma_{k-1} u(t) > 0 \qquad \text{for } t \in (t_{k-1}, t_k)\,,\ k = 1, 2,$$

where $t_0 := a$, $t_2 := b$.

Let $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ and t_1, t_2, ..., t_n are such that

$$a =: t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} := b. \tag{139}$$

Definition IV.2. We say that a vector-function $u = \operatorname{col}(u_1, u_2, \ldots, u_n) : [a, b] \to \mathbb{R}^n$ is of type $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ if every u_k, $k = 1, 2, \ldots, n$, is of type $(\sigma_{k0}, \sigma_{k1}; t_k)$.

We look for solutions $u = (u_i)_{i=1}^n$ of the boundary value problem (137), (138) such that each its component u_i is of type $(\sigma_{i0}, \sigma_{i1}; t_i)$ (i.e., vanishes at the point t_i and is of the sign σ_{i0} and σ_{i1}, respectively, on the left and on the right of it). The values of t_1, t_2, ..., t_n where the respective components vanish are not specified beforehand; they remain unknown and the problem is thus to find both u and t_1, t_2, ..., t_n.

The form of system (137) is motivated, in particular, by Emden-Fowler type equations and other equations where the non-linearity involves terms of type $[u]_\pm$ and $|u|$. From the computational point of view, these terms cause difficulties for the practical realisation of parametrisation schemes due to the need to integrate expressions depending on multiple parameters, and in such cases one should often complement the scheme with additional ingredients facilitating the computation (cf. § 10). In the cases where the components of solutions are expected to vanish at certain points, with a particular information available on the sign around it, complications of this kind can be avoided so that any additional approximation of integrands may not be needed [35].

§ 14. Parametrisation and reduction principle. Introduce some notation. Put

$$j_\sigma := \frac{1}{2}(\sigma + 1) \tag{140}$$

for any $\sigma \in \{-1, 1\}$ and define the function $\tilde{f} : [a, b] \times G \to \mathbb{R}^n$ by setting

$$\begin{aligned}\tilde{f}(t, u_1, \ldots, u_n) := f\big(t, &j_{\sigma_{11}} u_1, \ldots, j_{\sigma_{k-1,1}} u_{k-1}, j_{\sigma_{k0}} u_k, j_{\sigma_{k+1,0}} u_{k+1}, \ldots, j_{\sigma_{n0}} u_n, \\ &- j_{-\sigma_{11}} u_1, \ldots, -j_{-\sigma_{k-1,1}} u_{k-1}, -j_{-\sigma_{k0}} u_k, -j_{-\sigma_{k+1,0}} u_{k+1} \ldots, -j_{-\sigma_{n0}} u_n\big)\end{aligned} \tag{141}$$

for $u = (u_i)_i^n$ from G, $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n + 1$.

Proposition IV.3. *Let* $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ *be fixed. Then any* $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ *solution of* (137) *is a solution of the system*

$$u'(t) = \tilde{f}(t, u(t)), \qquad t \in [a, b], \tag{142}$$

where $\tilde{f}$ *is given by* (141). *Conversely, any* $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ *solution of* (142) *satisfies* (137).

Proposition IV.4. *For any* $u = (u_i)_i^n$ *is from* G, $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n + 1$,

$$\tilde{f}(t, u) = f\left(t, \frac{1}{2}(M_k + \mathbf{1}_n)\, u(t), \frac{1}{2}(M_k - \mathbf{1}_n)\, u(t)\right), \tag{143}$$

where

$$M_k := \operatorname{diag}(\sigma_{11}, \sigma_{21}, \ldots, \sigma_{k-1,1}, \sigma_{k0}, \sigma_{k+1,0}, \ldots, \sigma_{n0}). \tag{144}$$

When we are focusing on solutions described in Definition IV.2, these propositions allow us to easily rewrite the system in a simpler form (142) using the information known on the signs. Equality (143) implies, in particular, that when passing from (137) to (142), all the occurrences of $|u_i|$ in the original system are replaced by the ith component of $M_k u$ on $[t_{k-1}, t_k]$. The computation of the matrices M_k, $k = 1, \dots, n+1$, is straightforward.[30]

To treat the problem, we use parametrisation at multiple points [58] (§ 9) and replace the boundary value problem (142), (138) by a suitable family of auxiliary problems with separated conditions constructed as follows. We "freeze" the values of $u = \mathrm{col}(u_1, u_2, \dots, u_n)$ at points (139) by formally putting

$$u(t_k) = z^{(k)}, \qquad k = 0, 1, \dots, n+1, \tag{145}$$

where $z^{(k)} = \mathrm{col}\big(z_1^{(k)}, z_1^{(k)}, \dots, z_n^{(k)}\big)$, and consider the restrictions of system (142) to each of the intervals $[t_0, t_1]$, $[t_1, t_2]$, …, $[t_n, t_{n+1}]$. This leads us to the $n+1$ two-point boundary value problems on the respective subintervals

$$u'(t) = \tilde{f}(t, u(t)), \qquad t \in [t_{k-1}, t_k], \tag{146}$$

$$u(t_{k-1}) = z^{(k-1)}, \qquad u(t_k) = z^{(k)}, \tag{147}$$

where $k = 1, 2, \dots, n+1$ and $\tilde{f}$ is given by (141). Problems (146), (147) are then treated along the lines of [58]: we choose suitable $G_k \subset \mathbb{R}^n$, $k = 0, 1, \dots, n+1$ (p. 174), form the sets

$$G_{k-1,k} = \mathscr{C}(G_{k-1}, G_k), \qquad k = 1, 2, \dots, n+1, \tag{148}$$

and assume the following conditions:

1. There exist non-negative vectors $\varrho^{(1)}$, $\varrho^{(2)}, \dots, \varrho^{(n+1)}$ such that[31]

$$\varrho^{(k)} \geq \frac{t_k - t_{k-1}}{4} \delta_{[t_{k-1}, t_k], G_k(\varrho^{(k)})}(\tilde{f}) \tag{149}$$

   for all $k = 1, 2, \dots, n+1$.

2. There exist non-negative matrices K_1, K_2, …, K_{n+1} such that

$$f \in \mathrm{Lip}_{K_k}\big(G_k(\varrho^{(k)})\big), \qquad k = 1, 2, \dots, n+1. \tag{150}$$

3. The spectral radii of K_1, K_2, …, K_{n+1} satisfy the inequalities

$$r(K_k) < \frac{10}{3(t_k - t_{k-1})}, \qquad k = 1, 2, \dots, n+1. \tag{151}$$

Note that, although (149) and (151) both involve t_1, t_2, …, t_n, it makes sense to keep the present form of conditions (see § 15).

We define (p. 176) the iterations $u_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)$, $m = 0, 1, \dots$, by analogy with § 9, prove the existence of their uniform limits $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ as $m \to \infty$, and "glue" them together into the function $u_\infty(\cdot, z^{(0)}, z^{(1)}, z^{(2)}, \dots, z^{(n+1)}, t_1, \dots, t_n) : [a, b] \to \mathbb{R}^n$. This allows us to introduce the determining function $\Delta^{(k)} : G_{k-1} \times G_k \times (a, b)^2 \to \mathbb{R}^n$, $k = 1, \dots, n+1$,

$$\Delta^{(k)}(\xi, \eta, s_0, s_1) := \eta - \xi - \int_{t_{k-1}}^{t_k} \tilde{f}\big(s, u_\infty^{(k)}(s, \xi, \eta, s_0, s_1)\big) ds \tag{152}$$

for all $\xi \in G_{k-1}$, $\eta \in G_k$, and $\{s_0, s_1\} \subset (a, b)$.

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[30] For example, if $n = 3$ and we are interested in solutions of the type $[(1, -1; t_1), (-1, 1; t_2), (-1, 1; t_3)]$, then, according to (144), we have $M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $M_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $M_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $M_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

[31] In (149), we use the same notation as in (100), § 9: $G_k(\varrho) := \mathscr{O}_\varrho(G_{k-1,k})$, $k = 1, 2, \dots, n+1$, for any $\varrho \in \mathbb{R}_+^n$. We also use notation 6, p. 1.

Theorem IV.5. *The function* $u_\infty(\cdot, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n) : [a, b] \to \mathbb{R}^n$ *is a continuously differentiable solution of the boundary value problem* (137), (138) *if and only if the vectors* $z^{(k)}$, $k = 0, 1, 2, \ldots, n+1$, *and the points* $t_1, \ldots, t_n$ *satisfy the system of* $n(n+2)$ *numerical determining equations*

$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k) = 0, \qquad k = 1, 2, \ldots, n+1, \tag{153}$$

$$g\big(u_\infty^{(1)}(a, z^{(0)}, z^{(1)}, a, t_1), u_\infty^{(n+1)}(b, z^{(n)}, z^{(n+1)}, t_n, b)\big) = d. \tag{154}$$

Furthermore, this solution has type $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$.

Theorem IV.6. *If there exist some* $t_1, \ldots, t_n$ *from* (a, b) *and* $z^{(j)} \in G_j$, $j = 0, 1, \ldots, n+1$, *satisfying equations* (153), (154), *then the function*

$$u^*(t) = u_\infty\big(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n\big), \qquad t \in [a, b], \tag{155}$$

is a $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ *solution of the boundary value problem* (137), (138). *Conversely, if problem* (137), (138) *has a solution* $u^*(\cdot)$ *of type* $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$, *which, in addition, satisfies the conditions*

$$\begin{gathered} u^*(t_j) \in G_j, \qquad j = 0, 1, \ldots, n+1, \\ \{u^*(t) : t \in [t_{k-1}, t_k]\} \subset \mathscr{O}_{\varrho^{(k)}}(G_{k-1,k}), \qquad k = 1, 2, \ldots, n+1, \end{gathered} \tag{156}$$

then the system of determining equations (153), (154) *is satisfied with the same* $t_1, \ldots, t_n$, *and*

$$z^{(j)} := u^*(t_j), \qquad j = 0, 1, \ldots, n+1.$$

Moreover, the solution $u^*(\cdot)$ *necessarily has form* (155) *with these values of parameters.*

In this way, the question on determining $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ solutions of problem (137), (138) with the space localisation of graph (156) is reduced to finding the values $z^{(0)}, z^{(1)}, \ldots, z^{(n+1)}$, and $t_1, \ldots, t_n$ from equations (153), (154).

§ 15. Verification of conditions and practical realisation. Similarly to other cases where parametrisation is applied, the practical realisation of the scheme from § 14 uses approximate determining equations

$$\Delta_m^{(k)}(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k) = 0, \qquad k = 1, 2, \ldots, n+1, \tag{157}$$

$$g\big(u_m^{(1)}(a, z^{(0)}, z^{(1)}, a, t_1), u_m^{(n+1)}(b, z^{(n)}, z^{(n+1)}, t_n, b)\big) = d, \tag{158}$$

where, by analogy with (152), the functions $\Delta_m^{(k)} : G_{k-1} \times G_k \times (a, b)^2 \to \mathbb{R}^n$, $k = 1, \ldots, n+1$, are defined as

$$\Delta_m^{(k)}(\xi, \eta, s_0, s_1) := \eta - \xi - \int_{t_{k-1}}^{t_k} \tilde{f}\big(s, u_m^{(k)}(s, \xi, \eta, s_0, s_1)\big) ds \tag{159}$$

for $\xi \in G_{k-1}$, $\eta \in G_k$, and $\{s_0, s_1\} \subset (a, b)$. In contrast to (153), (154), for any fixed m, the mth approximate system (157), (158) contains only terms involving the functions $u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$, $k = 1, 2, \ldots, n+1$, which are computable explicitly.

In order to verify condition (149) on $\varrho^{(0)}, \ldots, \varrho^{(n+1)}$, it is needed to compute maximal and minimal values of the function $\tilde{f}$ over $\varrho^{(k)}$-neighbourhoods of sets $G_{k-1,k}$, $k = 1, 2, \ldots, n+1$, constructed according to (148). A relevant software can be used for this purpose (e.g., MAPLE, which has been applied in [35]). It is convenient to specify suitable sets

$$G^{(k)} \supset G_{k-1,k}, \qquad k = 1, 2, \ldots, n+1, \tag{160}$$

of simpler structure (e. g., parallelepipeds: if $G^{(k)}$ is a parallelepiped, then, by (6), the set $\mathscr{O}_{\varrho^{(k)}}(G^{(k)})$ is a parallelepiped as well) and use the inequality $\delta_{[\alpha,\beta],\tilde{G}}(\tilde{f}) \geq \delta_{[\alpha,\beta],G}(\tilde{f})$ for any $\tilde{G} \supset G$. Then the fulfilment of (149) is guaranteed if

$$\varrho^{(k)} \geq \frac{t_k - t_{k-1}}{4} \delta_{[t_{k-1},t_k],\mathscr{O}_{\varrho^{(k)}}(G^{(k)})}(\tilde{f}) \tag{161}$$

for $k = 1, 2, \ldots, n+1$. The same observation concerns the Lipschitz condition (150), which may be easier to check on the set $\mathscr{O}_{\varrho^{(k)}}(G^{(k)})$ instead of $\mathscr{O}_{\varrho^{(k)}}(G_{k-1,k})$, $k = 1, 2, \ldots, n+1$ (here, one can use software as well).

A natural question arises on the verification of the fulfilment of (149), (151), (161) due to the fact that the values $t_1, \ldots, t_n$ involved there are unknown.

Although, in (161) and (164), one can always majorise the lengths of the subintervals by $b - a$, this will lead to more restrictive conditions (and, possibly, to the need of introducing additional nodes in order to guarantee the convergence—the complication which could otherwise have been avoided). Another, better opportunity is to use preliminary results of computation. Thus, we start computations directly before checking conditions (149), (151). By doing so, we obtain a preliminary information on the space localisation of solutions and, as a consequence, a hint how to choose the regions where the conditions should be verified (see also the discussion in § 3). This concerns both the choice of the sets G_k, $k = 0, 1, \ldots, n+1$, with respect to the space variables and the intervals containing zeroes of solutions.

Suppose that, at some step of iteration (typically, already in the zeroth approximation), after solving approximate determining equations, we get certain approximations $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n$ of $t_1, t_2, \ldots, t_n$. If the computation shows signs of convergence, then these values are natural to be used to set restrictions of the form

$$T_k^- \leq t_k \leq T_k^+, \qquad k = 1, 2 \ldots, n, \tag{162}$$

by choosing appropriately the bounds T_k^-, T_k^+, $k = 1, 2, \ldots, n$. One can put in (162), e. g.,

$$T_k^- := \max\left\{a, \hat{t}_k - \frac{b-a}{n+1}\right\}, \qquad T_k^+ := \min\left\{\hat{t}_k + \frac{b-a}{n+1}, b\right\}$$

for $k = 1, 2, \ldots, n$ (however, finer estimates may be available in concrete cases). Knowing estimates of form (162), instead of (161), we can verify the conditions

$$\varrho^{(k)} \geq \frac{T_k^+ - T_{k-1}^-}{4} \delta_{[T_{k-1}^-,T_k^+],\mathscr{O}_{\varrho^{(k)}}(G^{(k)})}(\tilde{f}), \tag{163}$$

where $T_0^- = T_0^+ = a$, $T_{n+1}^- = T_{n+1}^+ = b$ and $G^{(k)}$, $k = 1, \ldots, n+1$, are suitably chosen parallelepipeds satisfying (160). In contrast to (161), the unknown time instants are no more involved in condition (163). Similarly, instead of (151), we will check the condition

$$r(K_k) < \frac{10}{3(T_k^+ - T_{k-1}^-)}, \tag{164}$$

where K_k is the Lipschitz matrix for the restriction of $\tilde{f}$ to $[t_{k-1}, t_k] \times \mathscr{O}_{\varrho^{(k)}}(G^{(k)})$, $k = 1, 2, \ldots, n+1$. Condition (164) is, of course, preferable to

$$\max_{1 \leq k \leq n+1} r(K_k) < \frac{10}{3(b-a)}, \tag{165}$$

and in the cases where (165) does not hold but (164) does, we can avoid unnecessary interval divisions which might otherwise be needed to construct a convergent scheme.

By proceeding in this manner we, strictly speaking, assume an additional condition (162) on the mutual disposition of the unknown nodes t_1, t_2, ..., t_n, which means that we consider the problem on $[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ solutions of (137), (138), (162). On the other hand, from the practical point of view, this is, in fact, no real restriction because, with a reasonable choice of bounds coming from numerical experiments, (162) simply means that we are not looking for the unknown values outside the intervals where their approximate values are already known to be contained. In other words, when applying this method to (137), (138) in concrete cases, we actually deal with (137), (138), (162), where (162) is added after some computation has been carried out.

An illustrative example showing the application of this technique can be found in [35] (see pp. 182–190 of this text). The computations have been carried out using MAPLE 14. We choose sets (160) and bounds (162) in condition (163) using the piecewise linear zeroth approximation (p. 186).

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Offprints of papers.

Offprints of the papers [36, 49, 56, 58, 60–63] are enclosed below.

[60] ‘Constructive analysis of periodic solutions with interval halving’ 40

[61] ‘Notes on interval halving procedure for periodic and two-point problems’ 74

[63] ‘A new approach to non-local boundary value problems for ordinary differential systems’ 94

[58] ‘On non-linear boundary value problems and parametrisation at multiple nodes’ ... 106

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[62] ‘Parametrisation for boundary value problems with transcendental non-linearities using polynomial interpolation’ 192

Boundary Value Problems
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Constructive analysis of periodic solutions with interval halving

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Abstract

For a constructive analysis of the periodic boundary value problem for systems of non-linear non-autonomous ordinary differential equations, a numerical-analytic approach is developed, which allows one to both study the solvability and construct approximations to the solution. An interval halving technique, by using which one can weaken significantly the conditions required to guarantee the convergence, is introduced. The main assumption on the equation is that the non-linearity is locally Lipschitzian.

An existence theorem based on properties of approximations is proved. A relation to Mawhin's continuation theorem is indicated.

MSC: 34B15

Keywords: periodic solution; periodic boundary value problem; parametrisation; periodic successive approximations; numerical-analytic method; interval halving; existence; continuation; Mawhin's theorem

Introduction

In this paper, we shall develop a numerical-analytic approach to the analysis of periodic solutions of systems of non-autonomous ordinary differential equations using the idea introduced in [1]. The method is numerical-analytic in the sense that its realisation consists of two stages concerning, respectively, an explicit construction of certain equations and their numerical analysis and is close in the spirit to the Lyapunov-Schmidt reductions [2, 3]. However, neither a small parameter nor an implicit function argument is used.

We focus on numerical-analytic schemes based upon successive approximations. In the context of the theory of non-linear oscillations, such types of methods were apparently first developed in [4–8]. We refer the reader to [9–20] for the related bibliography.

For a boundary value problem, the numerical-analytic approach usually replaces the problem by a family of initial value problems for a suitably perturbed system containing a vector parameter which most often has the meaning of the initial value of the solution. The solution of the Cauchy problem for the perturbed system is sought for in an analytic form by successive approximations, whereas the numerical value of the parameter is determined later from the so-called determining equations.

In order to guarantee the convergence, a kind of the Lipschitz condition is usually assumed [9–12] and a smallness restriction of the type

$$r(K) \leq q_T \tag{0.1}$$

is imposed, where K is the Lipschitz matrix and q_T depends on the period T. The improvement of condition (0.1) consists in maximising the value of the constant q_T.

In this paper, which is a continuation of [1], we provide a constructive approach to the study of solvability of the periodic problem (1.3), (1.4), where the analysis of convergence uses the interval halving technique. We shall see that, under fairly general assumptions, this idea allows one to replace (0.1) by the weaker condition

$$r(K) \leq 2q_T \tag{0.2}$$

and, thus, significantly improve the convergence conditions established, in particular, in [6–9, 12]. The restriction imposed on the width of the domain is likewise improved. Finally, an existence theorem based upon the properties of approximate solutions is proved. The proofs use a number of technical facts from [1], which are stated in the course of exposition when appropriate.

1 Problem setting and basic assumptions

The method that we are interested in deals with T-periodic solutions of a system of non-linear ordinary differential equations

$$u'(t) = f\big(t, u(t)\big), \quad t \in (-\infty, \infty), \tag{1.1}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a continuous function such that

$$f(t, z) = f(t + T, z) \tag{1.2}$$

for all $z \in \mathbb{R}^n$ and $t \in (-\infty, \infty)$. Here, T is a given positive number. We restrict ourselves to considering continuously differentiable solutions of system (1.1) and, furthermore, instead of T-periodic solutions of (1.1), we shall always deal with the solutions $u : [0, T] \to \mathbb{R}^n$ of the corresponding periodic boundary value problem on the bounded interval $[0, T]$,

$$u'(t) = f\big(t, u(t)\big), \quad t \in [0, T], \tag{1.3}$$

$$u(0) = u(T). \tag{1.4}$$

The passage to problem (1.3), (1.4) is justified by assumption (1.2).

Our main assumption is that $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitzian with respect to the space variable in a certain bounded set D, which is the closure of a bounded and connected domain in $\mathbb{R}^n$. For the sake of simplicity, we assume that there exists a non-negative constant square matrix K of dimension n such that

$$\big|f(t, x_1) - f(t, x_2)\big| \leq K|x_1 - x_2| \tag{1.5}$$

for all $\{x_1, x_2\} \subset D$ and $t \in [0, T]$.

Here and below, the obvious notation $|x| = \operatorname{col}(|x_1|, |x_2|, \ldots, |x_n|)$ is used, and the inequalities between vectors are understood componentwise. The same convention is adopted implicitly for the operations 'max' and 'min' so that, *e.g.*, $\max\{h(z) : z \in Q\}$ for any $h =$

$(h_i)_{i=1}^n : Q \to \mathbb{R}^n$, where $Q \subset \mathbb{R}^m$, $m \leq n$, is defined as the column vector with the components $\max\{h_i(z) : z \in Q\}$, $i = 1, 2, \dots, n$.

2 Notation and symbols

We fix an $n \in \mathbb{N}$ and a bounded set $D \subset \mathbb{R}^n$. The following symbols are used in the sequel:

1. 1_n is the unit matrix of dimension n.
2. $r(K)$ is the maximal, in modulus, eigenvalue of a matrix K.
3. Given a closed interval $J \subseteq \mathbb{R}$, we define the vector $\delta_{J,D}(f)$ by setting

$$\delta_{J,D}(f) := \max_{(t,z)\in J\times D} f(t,z) - \min_{(t,z)\in J\times D} f(t,z). \tag{2.1}$$

4. e_k, $k = 1, 2, \dots, n$: see (10.5).
5. $\partial\Omega$ is the boundary of a domain Ω.
6. $\rhd_S$: see Definition 10.1.

The notion of a set $D(r)$ associated with D, which could have been called an *inner r-neighbourhood* of D, will often be used in what follows.

Definition 2.1 For any non-negative vector $r \in \mathbb{R}^n$, we put

$$D(r) := \{z \in D : B(z,r) \subset D\}, \tag{2.2}$$

where

$$B(z,r) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq r\}. \tag{2.3}$$

One of the assumptions to be used below means that the inner r-neighbourhood of D is non-empty for r sufficiently large.

Finally, let the positive number ϱ_* be determined by the equality

$$\varrho_*^{-1} = \inf\left\{q > 0 : q^{-1} = \int_0^{\frac{1}{2}} \exp\bigl(\tau(\tau - 1)q\bigr)\, d\tau\right\}. \tag{2.4}$$

We refer, *e.g.*, to [12, 21] for the discussion of other ways of introducing the constant ϱ_* and for its meaning. What is important for us here is that ϱ_* is the constant appearing in Lemma 3.2. One can show by computation that

$$\varrho_* \approx 0.2927. \tag{2.5}$$

3 *p*-periodic successive approximations

The method suggested by Samoilenko in [6, 7], originally called *numerical-analytic method* for the investigation of periodic solutions, was also referred to later as the method of *periodic successive approximations* [9–12]. Its scheme, which is described in a suitable for us form by Propositions 3.1 and 3.4 below, is quite simple and deals with the investigation of the parametrised equation

$$u(t) = z + \int_0^t f\bigl(s, u(s)\bigr)\, ds - \frac{t}{T}\int_0^T f\bigl(s, u(s)\bigr)\, ds, \quad t \in [0, T], \tag{3.1}$$

where $z \in D$ is a parameter to be chosen later. For convenience of reference, we formulate the statements for the p-periodic problem

$$u'(t) = g(t, u(t)), \quad t \in [t_0, t_0 + p], \tag{3.2}$$

$$u(t_0) = u(t_0 + p), \tag{3.3}$$

where $g : [t_0, t_0 + p] \times \mathbb{R}^n \to \mathbb{R}^n$ and $t_0 \in (-\infty, \infty)$ is arbitrary but fixed.

Following [1], we now describe the original, unmodified, periodic successive approximations scheme for the p-periodic problem (3.2), (3.3) which we are going to modify and which is constructed as follows. With problem (3.2), (3.3), one associates the sequence of functions $u_m(\cdot, z)$, $m \geq 0$, defined according to the rule

$$\begin{aligned} u_0(t,z) &:= z, \\ u_m(t,z) &:= z + \int_{t_0}^{t} g(s, u_{m-1}(s,z))\,ds - \frac{t - t_0}{p} \int_{t_0}^{t_0+p} g(s, u_{m-1}(s,z))\,ds \end{aligned} \tag{3.4}$$

for $t \in [t_0, t_0 + p]$ and $m = 1, 2, \ldots$, where the vector $z = \operatorname{col}(z_1, z_2, \ldots, z_n)$ is regarded as a parameter, the value of which is to be determined later.

Proposition 3.1 ([12, Theorem 3.17]) *Let the function f satisfy the Lipschitz condition* (1.5) *with a matrix K for which the inequality*

$$r(K) < \frac{1}{p\varrho_*} \tag{3.5}$$

holds and, moreover,

$$D\left(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g)\right) \neq \emptyset. \tag{3.6}$$

Then, for any fixed $z \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$, the following assertions are true:

1. *Sequence* (3.4) *converges to a limit function*

$$u_\infty(t,z) = \lim_{m\to\infty} u_m(t,z) \tag{3.7}$$

uniformly in $t \in [t_0, t_0 + p]$.

2. *The limit function* (3.7) *satisfies the p-periodic boundary conditions*

$$u_\infty(t_0, z) = u_\infty(t_0 + p, z).$$

3. *The function $u_\infty(\cdot, z)$ is the unique solution of the Cauchy problem*

$$u'(t) = g(t, u(t)) - p^{-1}\Delta(z), \quad t \in [t_0, t_0 + p], \tag{3.8}$$

$$u(t_0) = z, \tag{3.9}$$

where

$$\Delta(z) := \int_{t_0}^{t_0+p} g(\tau, u_\infty(\tau, z))\,d\tau. \tag{3.10}$$

4. *Given an arbitrarily small positive* ε, *one can choose a number* $m_\varepsilon \geq 1$ *such that the estimate*

$$\left|u_m(t,z) - u_\infty(t,z)\right| \leq \frac{1}{2}\alpha_{m_\varepsilon}(t)K^{m_\varepsilon-1}(\varrho_\varepsilon pK)^{m-m_\varepsilon+1}(1_n - \varrho_\varepsilon pK)^{-1}\delta_{[t_0,t_0+p],D}(g)$$

holds for all $t \in [t_0, t_0 + p]$ *and* $m \geq m_\varepsilon$, *where*

$$\varrho_\varepsilon := \varrho_* + \varepsilon. \tag{3.11}$$

Recall that, according to (2.2), condition (3.6) means the non-emptiness of the inner $\frac{p}{4}\delta_{[t_0,t_0+p],D}(g)$-neighbourhood of the set D, where $\delta_{[t_0,t_0+p],D}(g)$ is the vector given by formula (2.1). This agrees with the natural supposition that, for an approximation technique to be applicable, the domain where the Lipschitz condition is assumed should be wide enough.

The proof of Proposition 3.1 is based on Lemma 3.2 formulated below, which provides an estimate for the sequence of functions α_m, $m \geq 0$, given by the formula

$$\alpha_m(t) := \left(1 - \frac{t-t_0}{p}\right)\int_{t_0}^{t} \alpha_{m-1}(s)\,ds + \frac{t-t_0}{p}\int_{t}^{t_0+p} \alpha_{m-1}(s)\,ds, \tag{3.12}$$

where $m \geq 1$ and $\alpha_0(t) := 1$, $t \in [t_0, t_0 + p]$. We provide the formulation here for a clearer understanding of the constants appearing in the estimates.

Lemma 3.2 ([16, Lemma 3]) *For any* $\varepsilon \in (0, +\infty)$, *one can specify an integer* $m_\varepsilon \geq 1$ *such that*

$$\alpha_{m+1}(t) \leq p(\varrho_* + \varepsilon)\alpha_m(t) \tag{3.13}$$

for all $t \in [t_0, t_0 + p]$ *and* $m \geq m_\varepsilon$.

It should be noted that estimate (3.13) is optimal in the sense that ε can never be put equal to zero.

Remark 3.3 It follows from [22, Lemma 4] that if $\varepsilon \geq \varepsilon_0$, where

$$\varepsilon_0 := \frac{3}{10} - \varrho_* \approx 0.00727, \tag{3.14}$$

then $m_\varepsilon = 2$ in Lemma 3.2 (here, of course, we think of m_ε as of the least integer possessing the property indicated).

The assertion of Proposition 3.1 suggests a natural way to establish a relation between the p-periodic solutions of the given equation (1.3) and those of the perturbed equation (3.8) (or, equivalently, solutions of the initial value problem (3.8), (3.9)). Indeed, it turns out that, by choosing the value of z appropriately, one can use function (3.7) to construct a solution of the original periodic boundary value problem (1.3), (1.4).

Proposition 3.4 ([1, 12]) *Let the assumptions of Proposition* 3.1 *hold. Then*:

1. *Given a* $z \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$, *the function* $u_\infty(\cdot, z)$ *is a solution of the p-periodic boundary value problem* (3.2), (3.3) *if and only if z is a root of the equation*

$$\Delta(z) = 0. \tag{3.15}$$

2. *For any solution* $u(\cdot)$ *of problem* (3.2), (3.3) *with* $u(t_0) \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$, *there exists a* z_0 *such that* $u(\cdot) = u_\infty(\cdot, z_0)$.

The important assertion (2) means that equation (3.15), usually referred to as a *determining equation*, allows one to track *all* the solutions of the periodic boundary value problem (1.3), (1.4). In such a manner, the original infinite-dimensional problem is reduced to a system of n numerical equations.

The method thus consists of two parts, namely, the *analytic* part, when the integral equation (3.1) is dealt with by using the method of successive approximations (3.4), and the *numerical* one, which consists in finding a value of the unknown parameter from equation (3.15). This closely correlates with the idea of the Lyapunov-Schmidt reduction [2, 3].

The main obstacle for an efficient application of Proposition 3.4 is due to the fact that the function $u_\infty(\cdot, z)$, $z \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$ and, therefore, the mapping $\Delta : D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g)) \to \mathbb{R}^n$ are explicitly unknown. Nevertheless, it is possible to prove the existence of a solution on the basis of the properties of a certain iteration $u_m(\cdot, z)$ which is constructed explicitly for a certain fixed m. For this purpose, one studies the *approximate determining system*

$$\Delta_m(z) = 0, \tag{3.16}$$

where $\Delta_m : D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g)) \to \mathbb{R}^n$ is defined by the formula

$$\Delta_m(z) := \int_{t_0}^{t_0+p} g\big(s, u_m(s, z)\big)\, ds$$

for $z \in D(\frac{p}{4}\delta_{[t_0,t_0+p],D}(g))$. This topic is discussed in detail, in particular, in [12], whereas a theorem of the kind specified, which corresponds to the scheme developed here, is proved in Section 9. Our main goal is to obtain a solvability theorem under assumptions weaker than those that would be needed when applying Proposition 3.1.

Indeed, in view of (2.5), assumption (3.5), which is essential for the proof of the uniform convergence of sequence (3.4), can be rewritten in the form

$$r(K) < p^{-1} 3.4161\ldots. \tag{3.17}$$

Inequality (3.17) can be treated either as a kind of upper bound for the Lipschitz matrix or as a smallness assumption on the period p, the latter interpretation presenting the scheme as particularly appropriate for the study of high-frequency oscillations.

Without assumption (3.17), Lemma 3.2 does not guarantee the convergence of sequence (3.4) when applied directly along the lines of the proof of Proposition 3.1. Nevertheless, it turns out that this limitation can be overcome and, by using a suitable parametrisation and modifying the scheme appropriately, one can always weaken the smallness condition (3.5) so that the constant on its right-hand side is *doubled*:

$$r(K) < \frac{2}{p\varrho_*}. \tag{3.18}$$

Note also that, although we have in mind to weaken mainly the smallness condition (3.17) guaranteeing the convergence of iterations, it turns out that the techniques suggested here for this purpose allow us to obtain a considerable improvement of condition (3.6) as well (Corollary 6.7).

Moreover, we shall see that, under the weaker condition (3.18), the modified scheme can be used to prove the existence of a periodic solution on the basis of results of computation (Theorem 10.2).

4 Interval halving, parametrisation and gluing

We should like to show that the approach described by Proposition 3.1 can also be used in the cases where the smallness condition (3.5), which guarantees the convergence, is violated. For this purpose, a natural trick based on the interval halving can be used, where the unmodified scheme, in a sense, should work twice. However, some care should be taken on the boundary conditions.

Indeed, from the first glance, one is tempted to implement halving in the sense that the original scheme should be applied for each of the resulting half-intervals, and thus sequence (3.4) would be constructed twice for problem (3.2), (3.3) with $t_0 = 0$, $p = \frac{1}{2}T$, $g = f|_{[0,\frac{1}{2}T]\times\mathbb{R}^n}$ and $t_0 = \frac{1}{2}T$, $p = \frac{1}{2}T$, $g = f|_{[\frac{1}{2}T,T]\times\mathbb{R}^n}$, respectively. This is impossible, however, because the boundary conditions on the half-intervals, with trivial exceptions, are never $\frac{1}{2}T$-periodic.

The correct halving scheme is obtained when, along with the periodic boundary value problem (1.3), (1.4), we consider two auxiliary problems

$$x'(t) = f\bigl(t, x(t)\bigr), \quad t \in \left[0, \frac{1}{2}T\right], \tag{4.1}$$

$$x\left(\frac{T}{2}\right) - x(0) = \lambda \tag{4.2}$$

and

$$y'(t) = f\bigl(t, y(t)\bigr), \quad t \in \left[\frac{1}{2}T, T\right], \tag{4.3}$$

$$y(T) - y\left(\frac{T}{2}\right) = -\lambda, \tag{4.4}$$

where $\lambda = \operatorname{col}(\lambda_1, \dots, \lambda_n)$ is a free parameter, the value of which is to be determined suitably from the argument related to gluing. The mutual disposition of the graphs of x and y satisfying, respectively, problems (4.1), (4.2) and (4.3), (4.4) is as shown on Figure 1.

Our further reasoning related to problem (1.3), (1.4) uses the following simple observation. Let us put

$$\chi_T(t) := \begin{cases} 1 & \text{for } 0 \le t < \frac{1}{2}T, \\ 0 & \text{for } \frac{1}{2}T \le t \le T. \end{cases} \tag{4.5}$$

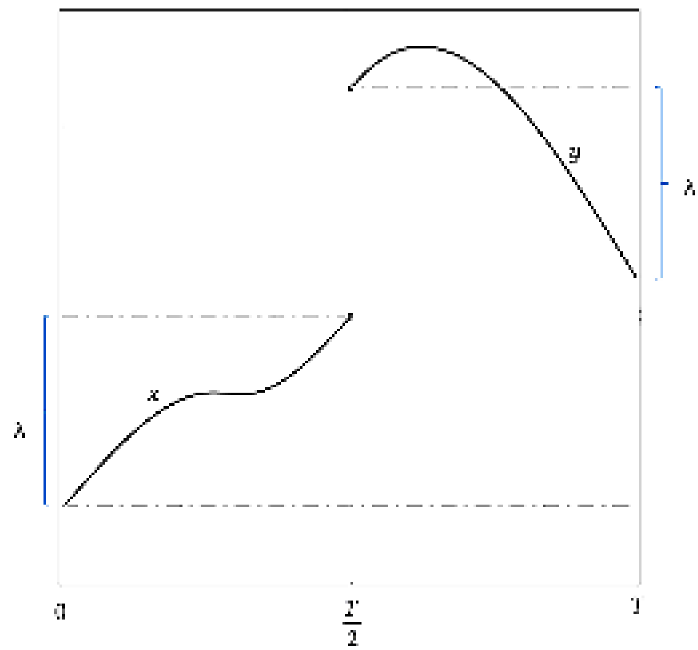

Figure 1 Solutions of auxiliary problems on half-intervals. Possible solutions of the auxiliary two-point problems (4.1), (4.2) and (4.3), (4.4) for arbitrary λ. The gluing means that, in case of solvability, λ is chosen so that (4.9) holds.

Proposition 4.1 ([1]) *Let* $x : [0, \frac{1}{2}T] \to \mathbb{R}^n$ *and* $y : [\frac{1}{2}T, T] \to \mathbb{R}^n$ *be solutions of problems* (4.1), (4.2) *and* (4.3), (4.4), *respectively, with a certain value of* $\lambda \in \mathbb{R}^n$. *Then the function*

$$u(t) := \chi_T(t)x(t) + \bigl(1 - \chi_T(t)\bigr)\left(y(t) - y\left(\frac{T}{2}\right) + x\left(\frac{T}{2}\right)\right), \quad t \in [0, T], \tag{4.6}$$

is a solution of the periodic problem boundary value problem (1.4) *for the equation*

$$u'(t) = f\left(t, u(t) + \bigl(1 - \chi_T(t)\bigr)\left(y\left(\frac{T}{2}\right) - x\left(\frac{T}{2}\right)\right)\right), \quad t \in [0, T]. \tag{4.7}$$

Conversely, if a certain function $u : [0, T] \to \mathbb{R}^n$ *is a solution of problem* (1.3), (1.4), *then its restrictions* $x := u|_{[0,\frac{1}{2}T]}$ *and* $y := u|_{[\frac{1}{2}T,T]}$ *to the corresponding intervals satisfy, respectively, problems* (4.1), (4.2) *and* (4.3), (4.4).

Remark 4.2 A solution of the functional differential equation (4.7) is understood in the Carathéodory sense, and a jump of u' at $\frac{1}{2}T$ is allowed. Note that function (4.6) is always continuous at $\frac{1}{2}T$.

The idea of Proposition 4.1 is, in fact, to rewrite the periodic boundary condition (1.4) in the form

$$u(0) - u\left(\frac{T}{2}\right) + u\left(\frac{T}{2}\right) - u(T) = 0, \tag{4.8}$$

which naturally leads us to the introduction of the parameter λ.

Proposition 4.1 allows one to treat the T-periodic problem (1.3), (1.4) as a kind of join of two independent two-point problems (4.1), (4.2) and (4.3), (4.4). Solving them independently and considering λ as an unknown parameter, one can then try to 'glue' their solutions together by choosing the value of λ so that (4.9) holds. The possibility of this gluing is equivalent to the solvability of the original problem. A rigorous formulation is contained in the following

Proposition 4.3 ([1]) *Assume that $x : [0, \frac{1}{2}T] \to \mathbb{R}^n$ and $y : [\frac{1}{2}T, T] \to \mathbb{R}^n$ are solutions of problems* (4.1), (4.2) *and* (4.3), (4.4), *respectively, for a certain value of $\lambda \in \mathbb{R}^n$. Then the function $u : [0, T] \to \mathbb{R}^n$ given by formula* (4.6) *is a solution of problem* (1.3), (1.4) *if and only if the equality*

$$x\left(\frac{T}{2}\right) = y\left(\frac{T}{2}\right) \tag{4.9}$$

holds.

Conversely, if a certain $u : [0, T] \to \mathbb{R}^n$ is a solution of problem (1.3), (1.4), *then the functions $x := u|_{[0,\frac{1}{2}T]}$ and $y := u|_{[\frac{1}{2}T,T]}$ satisfy, respectively, problems* (4.1), (4.2) *and* (4.3), (4.4).

Introduce the functions $\bar{\alpha}_m : [0, \frac{1}{2}T] \to [0, +\infty)$ and $\bar{\bar{\alpha}}_m : [\frac{1}{2}T, T] \to [0, +\infty)$, $m \geq 0$, by putting $\bar{\alpha}_0 \equiv 1$, $\bar{\bar{\alpha}}_0 \equiv 1$,

$$\bar{\alpha}_{m+1}(t) := \left(1 - \frac{2t}{T}\right)\int_0^t \bar{\alpha}_m(s)\,ds + \frac{2t}{T}\int_t^{\frac{1}{2}T} \bar{\alpha}_m(s)\,ds \tag{4.10}$$

for $t \in [0, \frac{1}{2}T]$, and

$$\bar{\bar{\alpha}}_{m+1}(t) := 2\left(1 - \frac{t}{T}\right)\int_{\frac{1}{2}T}^t \bar{\bar{\alpha}}_m(s)\,ds + \left(\frac{2t}{T} - 1\right)\int_t^T \bar{\bar{\alpha}}_m(s)\,ds \tag{4.11}$$

for $t \in [\frac{1}{2}T, T]$. In particular, we have

$$\bar{\alpha}_1(t) = 2t\left(1 - \frac{2t}{T}\right), \quad t \in \left[0, \frac{1}{2}T\right], \tag{4.12}$$

and

$$\bar{\bar{\alpha}}_1(t) = 2\left(1 - \frac{t}{T}\right)(2t - T), \quad t \in \left[\frac{1}{2}T, T\right]. \tag{4.13}$$

Functions (4.10) and (4.11), which are, in fact, appropriately scaled versions of (3.12), are involved in the estimates given in the sequel.

5 Iterations on half-intervals

As Proposition 4.3 suggests, our approach to the T-periodic problem (1.3), (1.4) requires that we first study the auxiliary problems (4.1), (4.2) and (4.3), (4.4) separately, for which purpose appropriate iteration processes will be introduced. Let us start by considering problem (4.1), (4.2). Following [1], we set

$$X_0(t, \xi, \lambda) := \xi + \frac{2t}{T}\lambda, \quad t \in \left[0, \frac{1}{2}T\right], \tag{5.1}$$

and define the recurrence sequence of functions $X_m : [0, \frac{1}{2}T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, $m = 0, 1, \ldots$, by putting

$$X_m(t,\xi,\lambda) := \int_0^t f\big(s, X_{m-1}(s,\xi,\lambda)\big)\,ds - \frac{2t}{T}\int_0^{\frac{T}{2}} f\big(s, X_{m-1}(s,\xi,\lambda)\big)\,ds + \xi + \frac{2t}{T}\lambda,$$
$$t \in \left[0, \frac{1}{2}T\right], \tag{5.2}$$

for all $m = 1, 2, \ldots$, $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$. In a similar manner, for the parametrised problem (4.3), (4.4) on the interval $[\frac{1}{2}T, T]$, we introduce the sequence of functions $Y_m : [\frac{1}{2}T, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, $m \geq 0$, according to the formulae

$$Y_0(t,\eta,\lambda) := \eta + \left(1 - \frac{2t}{T}\right)\lambda, \tag{5.3}$$

$$Y_m(t,\eta,\lambda) := \int_{\frac{T}{2}}^t f\big(s, Y_{m-1}(s,\eta,\lambda)\big)\,ds - \left(\frac{2t}{T} - 1\right)\int_{\frac{T}{2}}^T f\big(s, Y_{m-1}(s,\eta,\lambda)\big)\,ds$$
$$+ \eta + \left(1 - \frac{2t}{T}\right)\lambda, \quad t \in \left[\frac{1}{2}T, T\right], \tag{5.4}$$

for all η and λ from $\mathbb{R}^n$.

The recurrence sequences determined by equalities (5.1), (5.2) and (5.3), (5.4) arise in a natural way when boundary value problems of type (4.1), (4.1) and (4.3), (4.4) are considered. It is not difficult to verify that formulae (5.1), (5.2) and (5.3), (5.4) are particular cases of those corresponding the iteration scheme for two-point boundary value problems (see, *e.g.*, [23]). One can also derive these formulae directly from Proposition 3.1 by carrying out, respectively, the substitutions $x(t) = u(t) - 2tT^{-1}\lambda$, $t \in [0, \frac{1}{2}T]$, and $y(t) = u(t) + (2tT^{-1} - 1)\lambda$, $t \in [\frac{1}{2}T, T]$, after which one arrives at parametrised $\frac{1}{2}T$-periodic boundary value problems on the corresponding half-intervals.

It is important to note that all the members of the sequences $X_m(\cdot,\xi,\lambda)$, $m \geq 0$, and $Y_m(\cdot,\xi,\lambda)$, $m \geq 0$, satisfy, respectively, conditions (4.2) and (4.4).

Lemma 5.1 *For any* $\{\xi,\eta,\lambda\} \subset \mathbb{R}^n$ *and* $m \geq 0$, *the functions* $X_m(\cdot,\xi,\lambda)$ *and* $Y_m(\cdot,\eta,\lambda)$ *satisfy the boundary conditions*

$$X_m\left(\frac{T}{2},\xi,\lambda\right) - X_m(0,\xi,\lambda) = \lambda, \tag{5.5}$$

$$Y_m(T,\eta,\lambda) - Y_m\left(\frac{T}{2},\eta,\lambda\right) = -\lambda. \tag{5.6}$$

Now recall that the vector λ, which is involved in all the above-stated relations, is the 'gluing' parameter determining the pair of auxiliary boundary value problems (4.1), (4.2) and (4.3), (4.4), for which a continuous join described by Proposition 4.3 is possible. In this relation, the following property is important.

Lemma 5.2 *Let* $m \geq 0$ *be arbitrary. Then the equality*

$$X_m\left(\frac{T}{2},\xi,\lambda\right) = Y_m\left(\frac{T}{2},\eta,\lambda\right) \tag{5.7}$$

holds if and only if

$$\lambda = \eta - \xi. \tag{5.8}$$

Proof Indeed, it follows directly from (5.1) and (5.3) that $X_0(\frac{1}{2}T, \xi, \lambda) = \xi + \lambda$ and $Y_0(\frac{1}{2}T, \eta, \lambda) = \eta$, whence the assertion is obvious for $m = 0$. Similarly, if $m \geq 1$, then, according to (5.2) and (5.4), we have $X_m(\frac{1}{2}T, \xi, \lambda) = \xi + \lambda$ and $Y_m(\frac{1}{2}T, \eta, \lambda) = \eta$ and, consequently, relation (5.7) is equivalent to (5.8) for any m. □

6 Successive approximations and their convergence

Let us now pass to the construction of the iteration scheme for the original T-periodic problem (1.3), (1.4). The sequences $X_m : [0, \frac{1}{2}T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $Y_m : [\frac{1}{2}T, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$, $m \geq 0$, from the preceding section will be used for this purpose. We shall see that, for this purpose, the graphs of the respective members of the last named sequences should be glued together in the sense of Lemma 5.2. Namely, we put

$$x_m(t, \xi, \eta) := X_m(t, \xi, \eta - \xi), \tag{6.1}$$

$$y_m(t, \xi, \eta) := Y_m(t, \eta, \eta - \xi) \tag{6.2}$$

for any $m = 0, 1, \ldots$. Functions (6.1) and (6.2) will be considered only for those values of ξ and η that are located, in a sense, sufficiently far from the boundary of the domain D. More precisely, we consider (ξ, η) from the set $G_D(r)$, which, for any non-negative vector r, is defined by the equality

$$G_D(r) := \{(\xi, \eta) \in D^2 : B((1-\theta)\xi + \theta\eta, r) \subset D \text{ for all } \theta \in [0,1]\}. \tag{6.3}$$

Recall that we use notation (2.3). In other words, a couple of vectors (ξ, η) belongs to $G_D(r)$ if and only if every convex combination of ξ and η lies in D together with its r-neighbourhood. The inclusion $(\xi, \eta) \in G_D(r)$ implies, in particular, that $B(\xi, r) \subset D$ and $B(\eta, r) \subset D$, *i.e.*, the vectors ξ and η both belong to the set $D(r)$ defined by formula (2.2). It is also obvious from (6.3) that $G_D(r) \subset D^2$ for any r.

The following statement shows that sequence (6.1) is uniformly convergent and its limit is a solution of a certain perturbed problem for all (ξ, η) which are admissible in the sense that $(\xi, \eta) \in G_D(r)$ with r sufficiently large.

Theorem 6.1 *Let the vector-function* $f : [0, T] \times D \to \mathbb{R}^n$ *satisfy the Lipschitz condition* (1.5) *on the set D with a matrix K such that*

$$r(K) < \frac{2}{T\varrho_*}. \tag{6.4}$$

Moreover, assume that

$$G_D\left(\frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f)\right) \neq \varnothing. \tag{6.5}$$

Then, for an arbitrary pair of vectors $(\xi, \eta) \in G_D(\frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f))$:

1. *The uniform, in* $t \in [0, \frac{1}{2}T]$, *limit*

$$\lim_{m\to\infty} x_m(t,\xi,\eta) =: x_\infty(t,\xi,\eta) \tag{6.6}$$

exists and, moreover,

$$x_\infty\left(\frac{T}{2},\xi,\eta\right) - x_\infty(0,\xi,\eta) = \eta - \xi. \tag{6.7}$$

2. *The function* $x_\infty(\cdot,\xi,\eta)$ *is the unique solution of the Cauchy problem*

$$x'(t) = f\big(t,x(t)\big) + 2T^{-1}\Xi(\xi,\eta), \tag{6.8}$$

$$x(0) = \xi, \tag{6.9}$$

where

$$\Xi(\xi,\eta) := \eta - \xi - \int_0^{\frac{T}{2}} f\big(\tau, x_\infty(\tau,\xi,\eta)\big)\,d\tau. \tag{6.10}$$

3. *Given an arbitrarily small positive* ε, *one can specify a number* $m_\varepsilon \geq 1$ *such that*

$$\begin{aligned}&\big|x_m(\cdot,\xi,\eta) - x_\infty(\cdot,\xi,\eta)\big| \\ &\quad\leq \frac{1}{2}\bar{\alpha}_{m_\varepsilon}(t)K^{m_\varepsilon-1}\left(\frac{1}{2}T\varrho_\varepsilon K\right)^{m-m_\varepsilon+1}\left(1_n - \frac{1}{2}T\varrho_\varepsilon K\right)^{-1}\delta_{[0,\frac{1}{2}T],D}(f)\end{aligned} \tag{6.11}$$

for all $t \in [0, \frac{1}{2}T]$ *and* $m \geq m_\varepsilon$, *where* ϱ_ε *is given by* (3.11).

Recall that the constant ϱ_* involved in condition (6.4) is given by equality (2.4), while the vector $\delta_{[0,\frac{1}{2}T],D}(f)$ arising in (6.5) is defined according to (2.1).

Remark 6.2 The error estimate (6.11) may look inconvenient because it is guaranteed starting from a sufficiently large iteration number, m_ε, depending on the value of ε which can be arbitrarily small. It is, however, quite transparent when the required constant is not 'too close' to ϱ_* (*i.e.*, if ε is not 'too small'). More precisely, in view of Remark 3.3, $m_\varepsilon = 2$ for $\varepsilon \geq \varepsilon_0$, where

$$\varepsilon_0 \approx 0.00727$$

is given by formula (3.14). Consequently, inequality (6.11) with $\varepsilon \geq \varepsilon_0$ holds for an arbitrary value of $m \geq 2$.

By analogy with Theorem 6.1, under similar conditions, we can establish the uniform convergence of sequence (6.2). Namely, the following statement holds.

Theorem 6.3 *Assume that the vector-function f satisfies conditions* (1.5), (6.4) *and, moreover,*

$$G_D\left(\frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f)\right) \neq \varnothing. \tag{6.12}$$

Then, for all fixed $(\xi,\eta) \in G_D(\frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f))$:

1. *The uniform, in* $t \in [\frac{1}{2}T, T]$, *limit*

$$\lim_{m\to\infty} y_m(t,\xi,\eta) =: y_\infty(t,\xi,\eta) \tag{6.13}$$

exists and, moreover,

$$y_\infty(T,\xi,\eta) - y_\infty\left(\frac{T}{2},\xi,\eta\right) = \xi - \eta. \tag{6.14}$$

2. *The function* $y_\infty(\cdot,\xi,\eta)$ *is the unique solution of the Cauchy problem*

$$y'(t) = f\bigl(t, y(t)\bigr) + 2T^{-1}\mathrm{H}(\xi,\eta), \tag{6.15}$$

$$y\left(\frac{T}{2}\right) = \eta, \tag{6.16}$$

where

$$\mathrm{H}(\xi,\eta) := \xi - \eta - \int_{\frac{T}{2}}^{T} f\bigl(\tau, y_\infty(\tau,\xi,\eta)\bigr)\,d\tau. \tag{6.17}$$

3. *For an arbitrarily small positive* ε, *one can find a number* $m_\varepsilon \geq 1$ *such that*

$$\begin{aligned} &\bigl|y_m(\cdot,\xi,\eta) - y_\infty(t,\xi,\eta)\bigr| \\ &\quad \leq \frac{1}{2}\bar{\bar{\alpha}}_{m_\varepsilon}(t)K^{m_\varepsilon-1}\left(\frac{1}{2}T\varrho_\varepsilon K\right)^{m-m_\varepsilon+1}\left(1_n - \frac{1}{2}T\varrho_\varepsilon K\right)^{-1}\delta_{[\frac{1}{2}T,T],D}(f) \end{aligned} \tag{6.18}$$

for all $t \in [\frac{1}{2}T, T]$ *and* $m \geq m_\varepsilon$, *where* ϱ_ε *is given by* (3.11).

Remark 6.4 Similarly to Remark 6.2, one can conclude that the validity of estimate (6.18) is ensured for *all* $m \geq 1$ provided that $\varepsilon \geq \varepsilon_0$ with ε_0 given by formula (3.14).

Theorems 6.1 and 6.3 are improved versions of Theorems 1 and 2 from [1], and their proofs follow the lines of those given therein. The main difference here is the use of Lemma 7.2 in order to guarantee that the values of the iterations do not escape from D. The rest of the argument is pretty similar to that of [1], and we omit it.

Note that the assumptions of Theorems 6.1 and 6.3 differ from each other in conditions (6.5) and (6.12) only. Therefore, by putting

$$r_D(f) := \frac{T}{8}\max\bigl\{\delta_{[0,\frac{1}{2}T],D}(f), \delta_{[\frac{1}{2}T,T],D}(f)\bigr\}, \tag{6.19}$$

we arrive immediately at the following statement summarising the last two theorems.

Theorem 6.5 *Assume that the function* f *satisfies the Lipschitz condition* (1.5) *in* D *with* K *satisfying relation* (6.4) *and, moreover,* D *is such that*

$$G_D\bigl(r_D(f)\bigr) \neq \varnothing. \tag{6.20}$$

Then, for any $(\xi,\eta) \in G_D(r_D(f))$, *the assertions of Theorems* 6.1 *and* 6.3 *hold.*

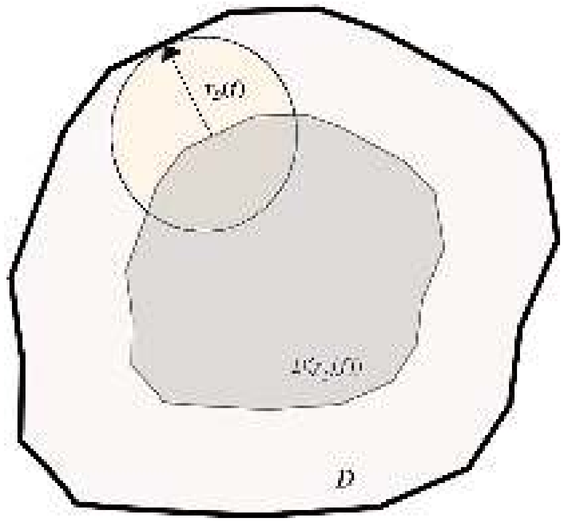

Figure 2 The set $D(r_D(f))$. A schematic picture of the set $D(r_D(f))$ appearing in condition (6.22) with $r_D(f)$ given by equality (6.19).

Recall that D is the main domain where the Lipschitz condition (1.5) is assumed, whereas $G_D(r_D(f))$ is the subset of D^2 defined according to (6.3). The latter set is, in a sense, a two-dimensional analogue of $D(r_D(f))$ and, as has already been noted above, the inclusion

$$G_D\big(r_D(f)\big) \subset D\big(r_D(f)\big) \times D\big(r_D(f)\big) \tag{6.21}$$

is true. By virtue of (6.21), assumption (6.20) implies in particular that

$$D\big(r_D(f)\big) \neq \varnothing, \tag{6.22}$$

which is a condition of type (3.6) appearing in Proposition 3.1 (see Figure 2). It turns out that, in the case of a convex domain, condition (6.20) can always be replaced by (6.22). Indeed, the following statement holds.

Lemma 6.6 *If the domain D is convex, then the corresponding set $G_D(r_D(f))$ has the form*

$$G_D\big(r_D(f)\big) = D\big(r_D(f)\big) \times D\big(r_D(f)\big).$$

Proof In view of (6.21), it is sufficient to show that

$$G_D\big(r_D(f)\big) \supset D\big(r_D(f)\big) \times D\big(r_D(f)\big). \tag{6.23}$$

Indeed, let us put $r := r_D(f)$ (the assertion is, of course, true for any non-negative vector r, but the present formulation is sufficient for our purposes) and assume that, on the contrary, inclusion (6.23) does not hold. Then one can specify some ξ and η such that

$$\{\xi, \eta\} \subset D(r), \tag{6.24}$$

$$(\xi, \eta) \notin G_D(r). \tag{6.25}$$

According to definition (6.3), relation (6.25) means the existence of certain $\theta_0 \in [0,1]$ and $z \in \mathbb{R}^n$ such that

$$z \in B\big((1-\theta_0)\xi + \theta_0\eta, r\big) \setminus D. \tag{6.26}$$

Let us put $h := z - (1-\theta_0)\xi - \theta_0\eta$. Then, in view of (6.26), we have

$$|h| \le r. \tag{6.27}$$

Furthermore, it is obvious that

$$(1-\theta_0)(\xi + h) + \theta_0(\eta + h) = z \tag{6.28}$$

and, consequently, z is a convex combination of $\xi + h$ and $\eta + h$. By virtue of (2.2), (6.24) and (6.27), both vectors $\xi + h$ and $\eta + h$ belong to D and, therefore, so does z because (6.28) holds and the set D is convex. However, this contradicts relation (6.26). Thus, inclusion (6.23) holds, and our lemma is proved. □

By virtue of Lemma 6.6, the assertion of Theorem 6.5 for f Lipschitzian in a convex domain can be reformulated as follows.

Corollary 6.7 *Let f satisfy conditions* (1.5) *and* (6.4). *If, moreover, the domain D is convex and* (6.22) *holds, then, for any ξ and η from $D(r_D(f))$, all the assertions of Theorems* 6.1 *and* 6.3 *hold*.

The convexity assumption on D is rather natural and, in fact, the domain where the Lipschitz condition for the non-linearity is verified most frequently has the form of a ball (in our case, where the inequalities between vectors are understood componentwise, it is an n-dimensional rectangular parallelepiped).

We note that the smallness assumption (6.4), which guarantees the convergence of iterations in Corollary 6.7, is twice as weak as the corresponding condition (3.5) of Proposition 3.1:

$$r(K) < \frac{1}{T\varrho_*}. \tag{6.29}$$

Furthermore, it is rather interesting to observe that the condition on inner neighbourhoods also becomes less restrictive after the interval halving has been carried out. Indeed, it is clear from (2.1) and (6.19) that, for condition (6.22) of Corollary 6.7 to be satisfied, it would be sufficient if

$$D\left(\frac{T}{8}\delta_{[0,T],D}(f)\right) \ne \varnothing, \tag{6.30}$$

whereas, at the same time, assumption (3.6) of Proposition 3.1 would require the relation

$$D\left(\frac{T}{4}\delta_{[0,T],D}(f)\right) \ne \varnothing. \tag{6.31}$$

The radius of the inner neighbourhood in (6.30) is less by half. Comparing (6.4) and (6.30) with the corresponding conditions (6.29) and (6.31) arising in Proposition 3.1, we conclude that the idea of interval halving described above thus allows us to improve the original scheme of periodic successive approximations in both directions.

Theorem 6.5 suggests that the iteration sequences (5.2) and (5.4) can be used to construct the solutions of auxiliary problems (4.1), (4.2) and (4.3), (4.4) and ultimately of the original problem (1.3), (1.4). A further analysis, which will lead us to an existence theorem, involves determining equations. Before continuing, we give some auxiliary statements.

7 Auxiliary statements

Several technical lemmata given below are needed in the proof of Theorems 6.1 and 6.3. We implicitly assume in the formulations that condition (6.20) is satisfied.

Given arbitrary $i \in \{0,1\}$ and $v \in C([\frac{1}{2}iT, \frac{1}{2}(i+1)T], \mathbb{R}^n)$, put

$$(P_i v)(t) := \int_{\frac{i}{2}T}^{t} v(s)\,ds - \left(\frac{2t}{T} - i\right) \int_{\frac{i}{2}T}^{\frac{i+1}{2}T} v(s)\,ds \tag{7.1}$$

for all $t \in [\frac{1}{2}iT, \frac{1}{2}(i+1)T]$. The linear mapping P_i, which obviously transforms the space $C([\frac{1}{2}iT, \frac{1}{2}(i+1)T], \mathbb{R}^n)$ to itself, is in fact a scaled version of the corresponding projection operator used rather frequently in studies of the periodic boundary problem (see, *e.g.*, [12]). In our case, properties of this mapping are used when estimating the values of the Nemytskii operator generated by the function f involved in equation (1.3).

Lemma 7.1 *Let $x : [0, \frac{1}{2}T] \to \mathbb{R}^n$ and $y : [\frac{1}{2}T, T] \to \mathbb{R}^n$ be arbitrary functions such that $\{x(t) : t \in [0, \frac{1}{2}T]\} \subset D$ and $\{y(t) : t \in [\frac{1}{2}T, T]\} \subset D$. Then*:

1. *For $t \in [0, \frac{1}{2}T]$*,

$$\begin{aligned} \left|P_0 f\left(\cdot, x(\cdot)\right)\right|(t) &\le \frac{1}{2}\bar{\alpha}_1(t)\delta_{[0,\frac{1}{2}T],D}(f) \\ &\le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f). \end{aligned} \tag{7.2}$$

2. *For $t \in [\frac{1}{2}T, T]$*,

$$\begin{aligned} \left|P_1 f\left(\cdot, y(\cdot)\right)\right|(t) &\le \frac{1}{2}\bar{\bar{\alpha}}_1(t)\delta_{[\frac{1}{2}T,T],D}(f) \\ &\le \frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f). \end{aligned} \tag{7.3}$$

Recall that $\bar{\alpha}_1$ and $\bar{\bar{\alpha}}_1$ are functions (4.12), (4.13), and the vectors $\delta_{[0,\frac{1}{2}T],D}(f)$, $\delta_{[\frac{1}{2}T,T],D}(f)$ are defined according to (2.1). The proof of Lemma 7.1 is almost a literal repetition of that of [1, Lemma 7] and uses the estimate obtained in [22, Lemma 3].

Lemma 7.2 *For arbitrary $m \ge 0$ and $(\xi, \eta) \in G_D(r_D(f))$, the inclusions*

$$\left\{x_m(t,\xi,\eta) : t \in \left[0, \frac{1}{2}T\right]\right\} \subset D \tag{7.4}$$

and

$$\left\{y_m(t,\xi,\eta) : t \in \left[\frac{1}{2}T, T\right]\right\} \subset D \tag{7.5}$$

hold.

Proof Let us fix an arbitrary pair of vectors

$$(\xi,\eta) \in G_D\big(r_D(f)\big) \tag{7.6}$$

and prove, *e.g.*, relation (7.4). We shall argue by induction. Indeed, in view of (5.1),

$$X_0(t,\xi,\eta-\xi) = \xi + \frac{2t}{T}(\eta-\xi) = \left(1-\frac{2t}{T}\right)\xi + \frac{2t}{T}\eta \tag{7.7}$$

for $t \in [0, \frac{1}{2}T]$. This means that, at every point t from $[0, \frac{1}{2}T]$, the value of $x_0(t,\xi,\eta)$ is a convex combination of ξ and η. Recalling definition (6.3) of the set $G_D(r_D(f))$ and using assumption (7.6), we conclude that all the values of the function $X_0(\cdot,\xi,\eta-\xi)$ lie in D, *i.e.*, (7.4) holds with $m = 0$.

Assume now that

$$\left\{X_m(t,\xi,\eta-\xi) : t \in \left[0, \frac{1}{2}T\right]\right\} \subset D \tag{7.8}$$

for a certain value of m and show that the inclusion

$$\left\{X_{m+1}(t,\xi,\eta-\xi) : t \in \left[0, \frac{1}{2}T\right]\right\} \subset D \tag{7.9}$$

holds as well. Indeed, considering (5.2) and recalling notation (7.1), we conclude that, for all m, the identity

$$\begin{aligned} P_0 f\big(\cdot, X_m(\cdot,\xi,\eta-\xi)\big)(t) &= X_{m+1}(t,\xi,\eta-\xi) - \xi - \frac{2t}{T}(\eta-\xi) \\ &= X_{m+1}(t,\xi,\eta-\xi) - \left(1-\frac{2t}{T}\right)\xi - \frac{2t}{T}\eta \end{aligned} \tag{7.10}$$

holds for any $t \in [0, \frac{1}{2}T]$. Since the validity of inclusion (7.4) has been assumed, we see that inequality (7.2) of Lemma 7.1 can be applied and, therefore, identity (7.10) yields

$$\left|X_{m+1}(t,\xi,\eta-\xi) - \left(1-\frac{2t}{T}\right)\xi - \frac{2t}{T}\eta\right| \le \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f) \tag{7.11}$$

for all $t \in [0, \frac{1}{2}T]$. It follows from (7.11) that, at every point $t \in [0, \frac{1}{2}T]$, the value $X_{m+1}(t,\xi,\eta-\xi)$ lies in the $\frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f)$-neighbourhood of a convex combination of the vectors ξ and η. Since ξ and η satisfy (7.6) and, by (6.19), $r_D(f) \ge \frac{T}{8}\delta_{[0,\frac{1}{2}T],D}(f)$, it follows from definition (6.3) of the set $G_D(r_D(f))$ that all the values of the function $X_{m+1}(\cdot,\xi,\eta-\xi)$ belong to D, *i.e.*, (7.9) holds. Thus, inclusion (7.4) is true for all $m \ge 0$. Recalling notation (6.1), we arrive immediately to (7.4).

Relation (7.5) is proved by analogy. Indeed, it follows from (5.3) that

$$\begin{aligned} Y_0(t,\eta,\eta-\xi) &= \eta + \left(1-\frac{2t}{T}\right)(\eta-\xi) = \left(\frac{2t}{T}-1\right)\xi + 2\left(1-\frac{t}{T}\right)\eta \\ &= \bigl(1-\theta(t)\bigr)\xi + \theta(t)\eta, \end{aligned} \tag{7.12}$$

where $\theta(t) := 2(1 - tT^{-1})$ for any $t \in [\frac{1}{2}T, T]$. Since, obviously, $0 \le \theta(t) \le 1$ for all $t \in [\frac{1}{2}T, T]$, identity (7.12) and assumption (7.6) guarantee that the function $Y_0(\cdot,\eta,\eta-\xi)$ has values in D. Let us assume that, for a certain m,

$$\left\{Y_m(t,\eta,\eta-\xi) : t \in \left[\frac{1}{2}T, T\right]\right\} \subset D \tag{7.13}$$

and show that

$$\left\{Y_{m+1}(t,\eta,\eta-\xi) : t \in \left[\frac{1}{2}T, T\right]\right\} \subset D. \tag{7.14}$$

By virtue of (5.4), for any $t \in [\frac{1}{2}T, T]$, we have

$$\begin{aligned} &P_1 f\bigl(\cdot, Y_m(\cdot,\eta,\eta-\xi)\bigr)(t) \\ &\quad = Y_{m+1}(t,\eta,\eta-\xi) - \eta - \left(1-\frac{2t}{T}\right)(\eta-\xi) \\ &\quad = Y_{m+1}(t,\eta,\eta-\xi) - \left(\frac{2t}{T}-1\right)\xi - 2\left(1-\frac{t}{T}\right)\eta \\ &\quad = Y_{m+1}(t,\eta,\eta-\xi) - \bigl(1-\theta(t)\bigr)\xi - \theta(t)\eta \end{aligned} \tag{7.15}$$

with the same definition of $\theta(\cdot)$ as in (7.12). According to assumption (7.13), the function $Y_m(\cdot,\xi,\eta-\xi)$ has values in D. Therefore, using equality (7.15) and estimate (7.3) of Lemma 7.1, we obtain

$$\bigl|Y_{m+1}(t,\eta,\eta-\xi) - \bigl(1-\theta(t)\bigr)\xi - \theta(t)\eta\bigr| \le \frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f) \tag{7.16}$$

for all $t \in [\frac{1}{2}T, T]$. Since $\theta : [\frac{1}{2}T, T] \to [0,1]$, inequality (7.16) implies that all the values of the function $Y_{m+1}(\cdot,\eta,\eta-\xi)$ belong to the $\frac{T}{8}\delta_{[\frac{1}{2}T,T],D}(f)$-neighbourhood of a convex combination of ξ and η. Recalling now (6.3) and (6.19) and using assumption (7.6), we arrive at (7.14). Consequently, inclusion (7.13) holds for all m, and (7.5) follows immediately from (6.2) and (7.13). The lemma is proved. □

Finally, the corresponding assertions of Theorems 6.1 and 6.3 lead us immediately to the following statement.

Lemma 7.3 *Under the assumptions of Theorem* 6.5, *the inclusions*

$$\left\{x_\infty(t,\xi,\eta) : t \in \left[0, \frac{1}{2}T\right]\right\} \subset D \tag{7.17}$$

and

$$\left\{y_\infty(t,\xi,\eta) : t \in \left[\frac{1}{2}T, T\right]\right\} \subset D \tag{7.18}$$

hold true for any $(\xi,\eta) \in G_D(r_D(f))$.

The proof of Lemma 7.3 consists in passing to the limit in (7.4) and (7.5) as $m \to +\infty$, the possibility of which is ensured by Theorem 6.5.

8 Limit functions and determining equations

The techniques based on the original periodic successive approximations (3.4), the applicability of which is guaranteed by Proposition 3.1, lead one to the necessary and sufficient conditions for the solvability formulated in terms of determining equations (3.15) of Proposition 3.4. A certain analogue of the last mentioned statement should also be established for our new version of the method, with iterations constructed using the interval halving procedure, for the resulting scheme to be logically complete. It is natural to expect that the limit functions of the iterations on the half-intervals will help one to formulate criteria of solvability of the original problem, and, in fact, it turns out that it is the functions $\Xi : G_D(r_D(f)) \to \mathbb{R}^n$ and $\mathrm{H} : G_D(r_D(f)) \to \mathbb{R}^n$ defined according to equalities (6.10) and (6.17) that provide such a characterisation.

Indeed, Theorems 6.1 and 6.3 guarantee that, under the conditions assumed, the functions $x_\infty(\cdot,\xi,\eta) : [0,\frac{1}{2}T] \to \mathbb{R}^n$ and $y_\infty(\cdot,\eta,\eta) : [\frac{1}{2}T,T] \to \mathbb{R}^n$ are well defined for all $(\xi,\eta) \in G_D(r_D(f))$. Therefore, by putting

$$\begin{aligned} u_\infty(t,\xi,\eta) &:= \chi_T(t)x_\infty(t,\xi,\eta) \\ &\quad + \bigl(1-\chi_T(t)\bigr)\left(y_\infty(t,\xi,\eta) - y_\infty\left(\frac{T}{2},\xi,\eta\right) + x_\infty\left(\frac{T}{2},\xi,\eta\right)\right), \\ &\quad y \in [0,T], \end{aligned} \tag{8.1}$$

we obtain a function $u_\infty(\cdot,\xi,\eta) : [0,T] \to \mathbb{R}^n$, which is well defined for the same values of $(\xi,\eta) \in G_D(r_D(f))$. This function is obviously continuous.

The following theorem, which is a modified version of [1, Theorem 4], establishes a relation of this function to the original periodic problem (1.3), (1.4) in terms of the zeroes of Ξ and H.

Theorem 8.1 *Let f satisfy the Lipschitz condition* (1.5) *with a matrix K such that* (6.4) *holds. Furthermore, assume that D has property* (6.20). *Then*:

1. *The function* $u_\infty(\cdot,\xi,\eta) : [0,T] \to \mathbb{R}^n$ *defined by* (8.1) *is a solution of the periodic boundary value problem* (1.3), (1.4) *if and only if the pair* (ξ,η) *satisfies the system of* $2n$ *equations*

$$\begin{aligned} \Xi(\xi,\eta) &= 0, \\ \mathrm{H}(\xi,\eta) &= 0. \end{aligned} \tag{8.2}$$

2. *For every solution* $u(\cdot)$ *of problem* (1.3), (1.4) *with* $(u(0),u(\frac{1}{2}T)) \in G_D(r_D(f))$, *there exists a pair* (ξ_0,η_0) *such that* $u(\cdot) = u_\infty(\cdot,\xi_0,\eta_0)$.

Equations (8.2) are usually referred to as *determining* or *bifurcation equations* [3, 12] because their roots determine solutions of the original problem. The variables involved in system (8.2) admit a natural interpretation: ξ means the value of the solution at 0, whereas η is responsible for its value at $\frac{1}{2}T$. We can observe the main difference between the unmodified periodic successive approximations (Proposition 3.1) and a similar scheme obtained after the interval halving (Theorem 6.5): the convergence condition is twice as weak but, instead of n numerical equations (3.15) of Proposition 3.4, we need to solve $2n$ equations (8.2) of Theorem 8.1.

A constructive solvability analysis involves a natural concept of approximate determining equations, which is discussed below.

9 Approximate determining equations

Although Theorem 8.1 provides a theoretical answer to the question on the construction of a solution of the periodic problem (1.3), (1.4), its application faces difficulties due to the fact that the explicit form of the functions $\Xi : G_D(r_D(f)) \to \mathbb{R}^n$ and $\mathrm{H} : G_D(r_D(f)) \to \mathbb{R}^n$ appearing in (8.2) is usually unknown. This complication can be overcome by using the functions

$$\Xi_m(\xi,\eta) := \eta - \xi - \int_0^{\frac{T}{2}} f\big(\tau, x_m(\tau,\xi,\eta)\big)\,d\tau \tag{9.1}$$

and

$$\mathrm{H}_m(\eta,\lambda) := \xi - \eta - \int_{\frac{T}{2}}^{T} f\big(\tau, y_m(\tau,\xi,\eta)\big)\,d\tau \tag{9.2}$$

for a fixed m, which will lead one to the so-called *approximate determining equations*. More precisely, similarly to [12, 24], it can be shown that, under certain natural assumptions, one can replace the exact determining system (8.2) by its approximate analogue

$$\begin{aligned} &\Xi_m(\xi,\eta) = 0, \\ &\mathrm{H}_m(\xi,\eta) = 0. \end{aligned} \tag{9.3}$$

Note that, unlike system (8.2), the mth approximate determining system (9.3) contains only terms involving the functions $x_m : [0, \frac{1}{2}T] \times G_D(r_D(f)) \to \mathbb{R}^n$ and $y_m : [\frac{1}{2}T, T] \times G_D(r_D(f)) \to \mathbb{R}^n$ and, thus, known explicitly.

It is natural to expect that approximations to the unknown solution of (1.3), (1.4) can be obtained by using the function $u_m(\cdot,\xi,\eta) : [0,T] \to \mathbb{R}^n$,

$$\begin{aligned} u_m(t,\xi,\eta) &:= \chi_T(t) x_m(t,\xi,\eta) \\ &\quad + \big(1-\chi_T(t)\big)\left(y_m(t,\xi,\eta) - y_m\left(\frac{T}{2},\xi,\eta\right) + x_m\left(\frac{T}{2},\xi,\eta\right)\right), \end{aligned} \tag{9.4}$$

which is an 'approximate' version of (8.1) well defined for all $t \in [0,T]$ and $(\xi,\eta) \in G_D(r_D(f))$.

The piecewise character of the definition of function (9.4) does not affect the properties that a potential approximation obtained from it should possess. Indeed,

Proposition 9.1 *If ξ and η satisfy equations* (9.3) *for a certain m, then the function $u_{m+1}(\cdot,\xi,\eta)$ determined by equality* (9.4) *is continuously differentiable on* $[0,T]$.

Proof It follows immediately from (5.2), (5.4) and (9.4) that

$$x'_{m+1}\left(\frac{T}{2},\xi,\eta\right) = f\left(\frac{T}{2},x_m\left(\frac{T}{2},\xi,\eta\right)\right) - \frac{2}{T}\int_0^{\frac{T}{2}} f\big(s,x_m(s,\xi,\eta)\big)\,ds + \frac{2}{T}(\eta-\xi) \tag{9.5}$$

and

$$y'_{m+1}\left(\frac{T}{2},\xi,\eta\right) = f\left(\frac{T}{2},y_m\left(\frac{T}{2},\xi,\eta\right)\right) - \frac{2}{T}\int_{\frac{T}{2}}^{T} f\big(s,y_m(s,\xi,\eta)\big)\,ds - \frac{2}{T}(\eta-\xi). \tag{9.6}$$

Recall that, by virtue of (5.4) and (6.2),

$$y_m\left(\frac{T}{2},\xi,\eta\right) = \eta.$$

Then, in view of (9.1) and (9.2), it follows from (9.3), (9.5) and (9.6) that

$$x'_{m+1}\left(\frac{T}{2},\xi,\eta\right) = y'_{m+1}\left(\frac{T}{2},\xi,\eta\right)$$

and, therefore, $u'_{m+1}(\cdot,\xi,\eta)$ is continuous at $\frac{1}{2}T$. The continuous differentiability of the function $u_{m+1}(\cdot,\xi,\eta)$ at other points is obvious from its definition. □

In order to prove a statement on the solvability of problem (1.3), (1.4), we need some estimates of the functions $\Xi_m : G_D(r_D(f)) \to \mathbb{R}^n$ and $\mathrm{H}_m : G_D(r_D(f)) \to \mathbb{R}^n$, $m = 0,1,\ldots,$ defined by (9.1) and (9.2).

Lemma 9.2 *Assume that* (6.20) *holds. Let f satisfy the Lipschitz condition* (1.5) *with a matrix K such that*

$$r(K) \le \frac{20}{3T}. \tag{9.7}$$

Then the estimates

$$\left|\Xi(\xi,\eta) - \Xi_m(\xi,\eta)\right| \le \frac{5T}{18}\left(\frac{3}{20}TK\right)^{m+1}\left(1_n - \frac{3}{20}TK\right)^{-1}\delta_{[0,\frac{1}{2}T],D}(f) \tag{9.8}$$

and

$$\left|\mathrm{H}(\xi,\eta) - \mathrm{H}_m(\xi,\eta)\right| \le \frac{5T}{18}\left(\frac{3}{20}TK\right)^{m+1}\left(1_n - \frac{3}{20}TK\right)^{-1}\delta_{[\frac{1}{2}T,T],D}(f) \tag{9.9}$$

hold for any values of $(\xi,\eta) \in G_D(r_D(f))$ and $m \ge 2$.

Proof Let us fix arbitrary $(\xi,\eta)\in G_D(r_D(f))$ and $m\geq 2$. Recalling (6.10) and (9.1), we obtain

$$\begin{aligned}\left|\Xi(\xi,\eta)-\Xi_m(\xi,\eta)\right| &= \left|\int_0^{\frac{T}{2}}\left[f\left(t,x_\infty(t,\xi,\eta)\right)-f\left(t,x_m(t,\xi,\eta)\right)\right]dt\right| \\ &\leq \int_0^{\frac{T}{2}}\left|f\left(t,x_\infty(t,\xi,\eta)\right)-f\left(t,x_m(t,\xi,\eta)\right)\right|dt. \end{aligned}\tag{9.10}$$

By Lemma 7.3, the function $x_\infty(\cdot,\xi,\eta):[0,\frac{1}{2}T]\to\mathbb{R}^n$ has values in D and, therefore, the Lipschitz condition (1.5) can be used in (9.10). Then, applying estimate (6.11) of Theorem 6.1 with $\varepsilon=\varepsilon_0$, where $\varepsilon_0\approx 0.00727$ is given by (3.14), we obtain

$$\begin{aligned}&\left|\Xi(\xi,\eta)-\Xi_m(\xi,\eta)\right| \\ &\quad\leq K\int_0^{\frac{T}{2}}\left|x_\infty(t,\xi,\eta)-x_m(t,\xi,\eta)\right|dt \\ &\quad\leq \frac{1}{2}\int_0^{\frac{T}{2}}\bar{\alpha}_{m_{\varepsilon_0}}(t)\,dtK^{m_{\varepsilon_0}}\left(\frac{1}{2}T\varrho_{\varepsilon_0}K\right)^{m-m_{\varepsilon_0}+1}\left(1_n-\frac{1}{2}T\varrho_{\varepsilon_0}K\right)^{-1}\delta_{[0,\frac{1}{2}T],D}(f).\end{aligned}\tag{9.11}$$

Recall now that, in view of Remark 6.2 and relations (3.11) and (3.14), one has

$$m_{\varepsilon_0}=2,\qquad \varrho_{\varepsilon_0}=\frac{3}{10},\tag{9.12}$$

and, therefore, (9.11) can be rewritten in the form

$$\begin{aligned}&\left|\Xi(\xi,\eta)-\Xi_m(\xi,\eta)\right| \\ &\quad\leq \frac{1}{2}\int_0^{\frac{T}{2}}\bar{\alpha}_2(t)\,dtK^2\left(\frac{3}{20}TK\right)^{m-1}\left(1_n-\frac{3}{20}TK\right)^{-1}\delta_{[0,\frac{1}{2}T],D}(f).\end{aligned}\tag{9.13}$$

Furthermore, it follows from (4.12) and (4.10) that the function $\bar{\alpha}_2$ has the form

$$\bar{\alpha}_2(t)=\left(\frac{16}{3T^2}t^3-\frac{16}{3T}t^2+t+\frac{T}{6}\right)t,\quad t\in\left[0,\frac{1}{2}T\right],\tag{9.14}$$

whence we obtain by computation that

$$\int_0^{\frac{T}{2}}\bar{\alpha}_2(t)\,dt=\frac{T^3}{80}.\tag{9.15}$$

Considering (9.12) and (9.15), we find that inequality (9.11), in fact, means that

$$\left|\Xi(\xi,\eta)-\Xi_m(\xi,\eta)\right|\leq\frac{T^3}{160}K^2\left(\frac{3}{20}TK\right)^{m-1}\left(1_n-\frac{3}{20}TK\right)^{-1}\delta_{[0,\frac{1}{2}T],D}(f),\tag{9.16}$$

which estimate coincides with (9.8). Note that the invertibility of the matrix $1_n-\frac{3}{20}TK$ is guaranteed by condition (9.7).

In a similar manner, in order to establish (9.9), we use (6.17) and (9.1) to obtain the estimate

$$\begin{aligned}\left|\mathrm{H}(\xi,\eta)-\mathrm{H}_m(\xi,\eta)\right| &= \left|\int_{\frac{T}{2}}^{T}\left[f\left(t,y_\infty(t,\xi,\eta)\right)-f\left(t,y_m(t,\xi,\eta)\right)\right]dt\right| \\ &\le \int_{\frac{T}{2}}^{T}\left|f\left(t,y_\infty(t,\xi,\eta)\right)-f\left(t,y_m(t,\xi,\eta)\right)\right|dt. \qquad (9.17)\end{aligned}$$

Lemma 7.3 guarantees that all the values of the function $y_\infty(\cdot,\xi,\eta):[\frac{1}{2}T,T]\to\mathbb{R}^n$ lie in D and, therefore, the Lipschitz condition (1.5) can be used in (9.17). Estimate (6.18) of Theorem 6.1 applied with $\varepsilon=\varepsilon_0$ then yields

$$\begin{aligned}&\left|\mathrm{H}(\xi,\eta)-\mathrm{H}_m(\xi,\eta)\right| \\ &\quad\le K\int_{\frac{T}{2}}^{T}\left|y_\infty(t,\xi,\eta)-y_m(t,\xi,\eta)\right|dt \\ &\quad\le \frac{1}{2}\int_{\frac{T}{2}}^{T}\bar{\bar{\alpha}}_{m_{\varepsilon_0}}(t)\,dt K^{m_{\varepsilon_0}}\left(\frac{1}{2}T\varrho_{\varepsilon_0}K\right)^{m-m_{\varepsilon_0}+1}\left(1_n-\frac{1}{2}T\varrho_{\varepsilon_0}K\right)^{-1}\delta_{[\frac{1}{2}T,T],D}(f). \qquad (9.18)\end{aligned}$$

Finally, it follows from (4.13) and (4.11) by computation that

$$\bar{\bar{\alpha}}_1(t)=\frac{16}{3T^2}t^4-\frac{16}{T}t^3+17t^2-\frac{15T}{2}t+\frac{7T^2}{6},\quad t\in\left[\frac{1}{2}T,T\right], \qquad (9.19)$$

and, hence,

$$\int_{\frac{T}{2}}^{T}\bar{\bar{\alpha}}_2(t)\,dt=\frac{T^3}{80}. \qquad (9.20)$$

Consequently, by virtue of relations (9.12) and (9.20), inequality (9.18) leads us directly to the required estimate (9.9). □

10 Solvability analysis based on approximation

The argument shown above allows us to conclude on the solvability of the periodic problem (1.3), (1.4) on the basis of properties of iterations (5.2) and (5.4). More precisely, it turns out that the use of functions (9.1) and (9.2) allows one to study the vector field $\Phi: G_D(r_D(f))\to\mathbb{R}^{2n}$,

$$\Phi(\xi,\eta):=\begin{pmatrix}\eta-\xi-\int_0^{\frac{T}{2}}f(\tau,x_\infty(\tau,\xi,\eta))\,d\tau\\ \xi-\eta-\int_{\frac{T}{2}}^{T}f(\tau,y_\infty(\tau,\xi,\eta))\,d\tau\end{pmatrix},\quad (\xi,\eta)\in G_D\left(r_D(f)\right), \qquad (10.1)$$

the critical points of which, as we have seen in Theorem 8.1, determine the solutions of the original problem (1.3), (1.4), through its approximation

$$\Phi_m(\xi,\eta):=\begin{pmatrix}\eta-\xi-\int_0^{\frac{T}{2}}f(\tau,x_m(\tau,\xi,\eta))\,d\tau\\ \xi-\eta-\int_{\frac{T}{2}}^{T}f(\tau,y_m(\tau,\xi,\eta))\,d\tau\end{pmatrix},\quad (\xi,\eta)\in G_D\left(r_D(f)\right), \qquad (10.2)$$

where m is fixed. In the formulation of the theorem given below, the following notion is used.

Definition 10.1 ([12]) Let r and l be positive integers and $S \subset \mathbb{R}^l$ be an arbitrary non-empty set. For any pair of vector functions $g_j : \mathbb{R}^l \to \mathbb{R}^r$, $j = 1, 2$, we write

$$g_1 \rhd_S g_2 \tag{10.3}$$

if and only if there exists a function $\nu : S \to \{1, 2, \ldots, l\}$ such that the strict inequality

$$\langle g_1(z) - g_2(z), e_{\nu(z)} \rangle > 0 \tag{10.4}$$

holds for all $z \in S$.

Here, e_k, $k = 1, 2, \ldots, r$, are the unit vectors,

$$e_k := \operatorname{col}(\underbrace{0, 0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0), \tag{10.5}$$

and $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^r$. The binary relation $\rhd_S$ introduced by Definition 10.1 is a kind of strict inequality for vector functions and its properties are similar to those of the usual strict inequality sign. For example, $f \geq g$ and $g \rhd_S h$ imply that $f \rhd_S h$. The last named property will be used below in the proof of Theorem 10.2.

We are now able to formulate a statement guaranteeing the solvability of the original periodic problem (1.3), (1.4) based on the information obtained in the course of computation of iterations. In contrast to the unmodified scheme of periodic successive approximations (Proposition 3.1, $r(K) < T^{-1}\varrho_*^{-1}$), here the iterations are proved to be convergent under the assumption that is twice as weak as in the former case (Theorem 6.5, $r(K) < 2T^{-1}\varrho_*^{-1}$). A similar observation can be made concerning the assumption on the domain D (see Corollary 6.7 and the remarks related to conditions (6.30) and (6.31)).

When stating the existence theorem, we restrict our consideration to a slightly weaker version of condition (6.4), where the value $\varrho_* \approx 0.2927$ is replaced by 0.3, and thus neglect the gap $(0, 0.00727\ldots)$ for ε in estimates (6.11) and (6.18).

Theorem 10.2 *Assume that the function f in* (1.3) *satisfies the Lipschitz condition* (1.5) *with a matrix K such that inequality* (9.7) *holds and, moreover, the set D has property* (6.20). *Moreover, let there exist a closed domain*

$$\Omega \subset G_D\big(r_D(f)\big)$$

such that, for a certain fixed value of $m \geq 2$, the mapping Φ_m given by formula (10.2) *satisfies the conditions*

$$\deg(\Phi_m, \Omega) \neq 0 \tag{10.6}$$

and

$$|\Phi_m| \rhd_{\partial\Omega} \frac{5T}{18} \begin{pmatrix} M_m \delta_{[0, \frac{1}{2}T], D}(f) \\ M_m \delta_{[\frac{1}{2}T, T], D}(f) \end{pmatrix}, \tag{10.7}$$

where

$$M_m := \left(\frac{3}{20}TK\right)^{m+1}\left(1_n - \frac{3}{20}TK\right)^{-1}. \tag{10.8}$$

Then there exist certain values $(\xi^*, \eta^*) \in \Omega$ *such that the function* $u_\infty(\cdot, \xi^*, \eta^*)$ *is a solution of the periodic boundary value problem* (1.3), (1.4).

Recall that the symbol $\rhd_{\partial\Omega}$ in (10.7) is understood in the sense of Definition 10.1. It should be noted that condition (10.7) involves the values of functions on the boundary of Ω only.

Proof We shall use the lemmata stated above. By analogy to [12, 24], we shall prove that the fields Φ and Φ_m are homotopic. It will be sufficient to consider the linear deformation

$$Q_\theta := \Phi_m + \theta(\Phi - \Phi_m), \tag{10.9}$$

where $\theta \in [0,1]$. Indeed, it is clear that Q_θ is a continuous mapping on $\partial\Omega$ for every $\theta \in [0,1]$ and, furthermore,

$$Q_0 = \Phi_m, \qquad Q_1 = \Phi. \tag{10.10}$$

Let us fix an arbitrary pair $(\xi, \eta) \in \partial\Omega$. According to (10.1) and (10.2), we have

$$\begin{aligned} \left|Q_\theta(\xi,\eta)\right| &= \left|\Phi_m(\xi,\eta) + \theta\left[\Phi(\xi,\eta) - \Phi_m(\xi,\eta)\right]\right| \\ &\geq \left|\Phi_m(\xi,\eta)\right| - \left|\Phi(\xi,\eta) - \Phi_m(\xi,\eta)\right|. \end{aligned} \tag{10.11}$$

On the other hand, by Lemma 9.2, estimates (9.8) and (9.9) true. Using relations (9.8) and (9.9) in (10.11), we show that

$$|Q_\theta| \rhd_{\partial\Omega} 0$$

and hence Q_θ does not vanish on $\partial\Omega$ for any θ. Thus, Φ is homotopic to Φ_m. The property of invariance of degree under homotopy then yields

$$\deg(\Phi, \Omega) = \deg(\Phi_m, \Omega);$$

and therefore, in view of (10.6), we conclude that $\deg(\Phi, \Omega) \neq 0$. Consequently, there exist vectors ξ^* and η^* possessing the properties indicated, and it only remains to refer to Theorem 8.1. The theorem is proved. □

Note that Theorem 10.2 provides solvability conditions based upon properties of approximations starting from the second one inclusively. A similar statement allowing to use the zeroth and the first approximations can be obtained if we use [12, Lemma 3.16] instead of Lemma 3.2. In that case, condition (10.7) of Theorem 10.2 is replaced, respectively, by the relations

$$|\Phi_0| \rhd_{\partial\Omega} \frac{5T^2}{108}\begin{pmatrix} K(1_n - \frac{3}{20}TK)^{-1}\delta_{[0,\frac{1}{2}T],D}(f) \\ K(1_n - \frac{3}{20}TK)^{-1}\delta_{[\frac{1}{2}T,T],D}(f) \end{pmatrix} \tag{10.12}$$

and

$$|\Phi_1| \rhd_{\partial\Omega} \frac{T^3}{144} \begin{pmatrix} K^2(1_n - \frac{3}{20}TK)^{-1}\delta_{[0,\frac{1}{2}T],D}(f) \\ K^2(1_n - \frac{3}{20}TK)^{-1}\delta_{[\frac{1}{2}T,T],D}(f) \end{pmatrix}. \tag{10.13}$$

11 Approximation of a solution

The theorem proved in the preceding section can be complemented by the following natural observation. Let $(\hat{\xi}, \hat{\eta}) \in \Omega$ be a root of the approximate determining system (9.3) for a certain m. Then the function

$$U_m(t) := u_m(t, \hat{\xi}, \hat{\eta}), \quad t \in [0, T], \tag{11.1}$$

defined according to (9.4) can be regarded as the mth approximation to a solution of the periodic problem (1.3), (1.4). This is justified by Proposition 9.1 and the estimates

$$\left|x_\infty(t, \hat{\xi}, \hat{\eta}) - U_m(t)\right| \le \frac{1}{2}\bar{\alpha}_1(t)\left(\frac{3}{20}TK\right)^m \left(1_n - \frac{3}{20}TK\right)^{-1} \delta_{[0,\frac{1}{2}T],D}(f) \tag{11.2}$$

for $t \in [0, \frac{1}{2}T]$ and

$$\left|y_\infty(t, \hat{\xi}, \hat{\eta}) - U_m(t)\right| \le \frac{1}{2}\bar{\alpha}_1(t)\left(\frac{3}{20}TK\right)^m \left(1_n - \frac{3}{20}TK\right)^{-1} \delta_{[\frac{1}{2}T,T],D}(f) \tag{11.3}$$

for $t \in [\frac{1}{2}T, T]$, which, as is easy to see from (9.4), follow directly from Theorem 6.5. A uniform inequality, not given here, can be obtained by estimating the mapping $(\xi, \eta) \mapsto u_m(t, \xi, \eta)$ for any fixed $t \in [0, T]$.

It is worth to emphasise the role of the unknown parameters whose values appearing in (11.1) are determined from equations (9.3): $\hat{\xi}$ is an approximation of the initial value of the periodic solution and $\hat{\eta}$ is that of its value at $\frac{1}{2}T$.

As regards the practical application of Theorem 10.2, it should be noted that, according to (10.2), the mapping Φ_m is known in an analytic form because it is determined solely by the mth iteration, which is already constructed at the moment. Of course, the degree in (10.6) is the Brouwer degree because all the vector fields are finite-dimensional. Likewise, all the terms in the right-hand side of inequality (10.7) are computed explicitly (*e.g.*, by using computer algebra systems).

12 An example

Let us consider the scalar π-periodic boundary value problem

$$u'(t) = -u(t) - \frac{1}{2}\big(u(t)\big)^2 + h(t), \quad t \in [0, \pi], \tag{12.1}$$

$$x(0) = x(\pi), \tag{12.2}$$

where $h(t) := \frac{1}{2}(3\cos 2t - \sin 2t) + \frac{1}{8}(\sin 4t + 1)$, $t \in [0, \pi]$. It is easy to check that the function

$$u(t) = \frac{1}{2}(\cos 2t + \sin 2t), \quad t \in [0, \pi], \tag{12.3}$$

is a solution of problem (12.1), (12.2). This solution has values in the domain $D := [-1,1]$, where, as one can verify, the convergence condition (3.5) is not satisfied. However, the corresponding condition with the doubled constant (3.18) *does* hold, and therefore, the interval halving technique can be used.

The appropriate computations, which have been carried out by using *Maple* 14 and are omitted here, show that the approach based on Theorems 6.5 and 10.2 is indeed applicable in this case. The existence of solution (12.3) (let us forget for a moment that we know it explicitly in this academic example) is established by Theorem 10.2, whereas its approximations of type (11.1) are constructed as described above. For instance, in the first approximation, we have $u \approx U_1$ with

$$U_1(t) := \chi_\pi(t)x_1(t) + \bigl(1 - \chi_\pi(t)\bigr)y_1(t), \quad t \in [0,\pi],$$

where χ_π is the indicator function (4.5) and

$$\begin{aligned} x_1(t) :=\ & -0.0266088895 + 0.7499999998(\cos t)^2 + 1.499999999 \sin t \cos t \\ & - 0.25(\cos t)^4 - 0.4392877052t, \quad t \in \left[0, \frac{1}{2}T\right], \end{aligned} \tag{12.4}$$

and

$$\begin{aligned} y_1(t) :=\ & -1.406671916 + 0.07499999995(\cos t)^2 - 0.25(\cos t)^4 \\ & + 1.5 \sin t \cos t + 0.4392877052t, \quad t \in \left[\frac{1}{2}T, T\right]. \end{aligned} \tag{12.5}$$

The numerical values of the parameters ξ and η corresponding to functions (12.4), (12.5) (see Table 1) are found from the system of equations (9.3) with $m = 1$, which, in this case, have the form

$$\begin{aligned} & 2048\pi^2\xi^2\eta + \bigl(6144 - 15360\pi + 5824\pi^2 + 2048\pi^2\eta\bigr)\xi\eta \\ & \quad + \bigl(31488\pi + 905\pi^2 - 3072 + 33792\pi\eta - 6144\eta + 6848\pi^2\eta + 2048\pi^2\eta^2\bigr)\eta = 0 \end{aligned}$$

and

$$\begin{aligned} & 2048\pi^2\xi^2\eta + \bigl(15360\pi + 6144 + 5824\pi^2 + 2048\pi^2\eta\bigr)\xi\eta \\ & \quad + \bigl(905\pi^2 - 31488\pi - 3072 - 33792\pi\eta - 6144\eta + 6848\pi^2\eta + 2048\pi^2\eta^2\bigr)\eta = 0. \end{aligned}$$

Table 1 Approximate values of parameters at several steps of iteration for problem (12.1), (12.2). The last row corresponds to the exact solution (12.3)

Iteration	ξ	η
0	0.3586778912	–0.1413221085
1	0.4733911105	–0.7166404021
2	0.5053539028	–0.5122443553
3	0.5079847542	–0.4747166175
4	0.498648589	–0.5011705927
...	...	...
∞	0.5	–0.5

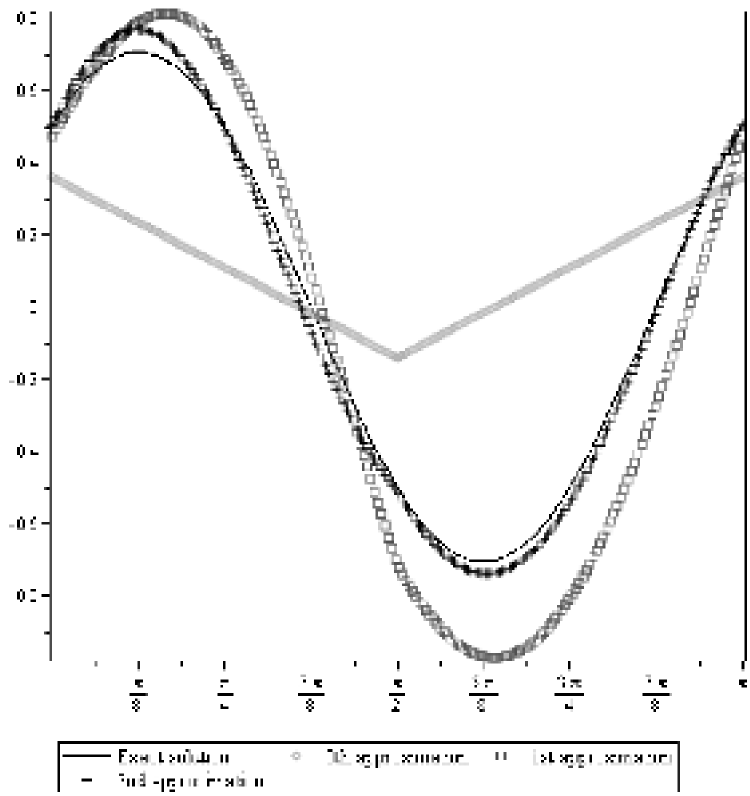

Figure 3 Solution (12.3) of problem (12.1), (12.2) and its 0th, 1st and 2nd approximations.

The graphs obtained in the course of computation are shown on Figures 3 and 4, whereas Table 1 contains the corresponding numerical values of the parameters. Note that only the zeroth approximation has derivative with a discontinuity at $\frac{1}{2}T$ (*cf.* Proposition 9.1). The graphs and the computed numerical values of the parameters show a rather good accuracy of approximation.

13 Comments

Several points can be outlined in relation to the techniques discussed in the preceding sections.

13.1 Approximation scheme in practice

An interesting feature of the approach indicated here is that a practical analysis of the periodic problem (1.3), (1.4) along its lines starts directly with the computation of iterations. We construct the approximate determining equations (9.3), solve them numerically in an appropriate region, substitute the corresponding roots into the formula for u_m and form functions (11.1) which are, in a sense, candidates for approximations of a solution. Having constructed functions (11.1) for several values of m, we check their behaviour heuristically and if it exhibits some signs of being possibly convergent, we stop the computation and verify the assumptions of the existence theorem. If successful, then, since this moment, we already know that a solution exists, and either we are satisfied with the achieved accuracy of approximation (in this case, the scheme stops and the function U_m given by (11.1) for the last computed value of m is proclaimed as its outcome) or, for some reasons, we find

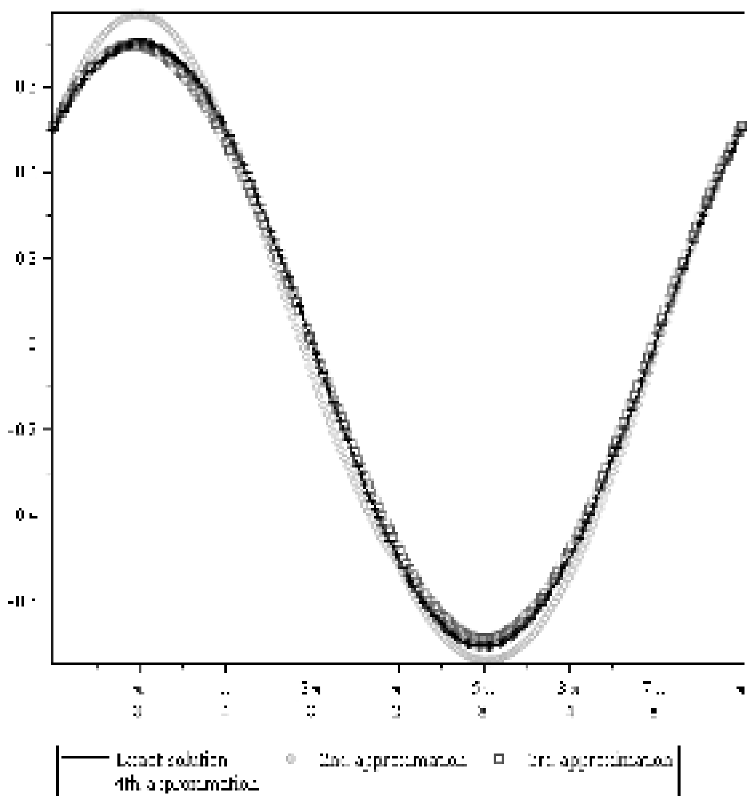

Figure 4 Solution (12.3) of problem (12.1), (12.2) and its 2nd, 3rd and 4th approximations.

that a more accurate approximation is needed (one more step is made then, and a similar check is carried out for the new approximation).

It is important to observe that, once the existence of a solution is known from Theorem 10.2 at the mth step of iteration, we *immediately* obtain an approximation to it in the form (11.1). The scheme thus allows us to both study the solvability of the periodic problem and construct approximations to its solution.

It should be noted that the ability to derive the fact of solvability of the original problem from the corresponding properties of approximate problems is rather uncommon (see [12] for some details). For the numerical methods, the generic situation is, in fact, quite the reverse, when some or another technique is applied to solve a problem which is *a priori* assumed to be solvable.

13.2 Extension to other problems

The idea expressed above can easily be adopted for application to differential equations with argument deviations. The only issue that should be clarified in that case is the definition of iterations on the half-intervals at those points which are thrown over the middle to the adjacent half-interval. For this purpose, sequences (5.2) and (5.4) should be computed simultaneously, with (5.4) serving as an initial function for (5.2) at the next step, and *vice versa*.

Likewise, with appropriate modifications, the technique developed here can be applied to problems with boundary conditions other than periodic ones. We do not dwell on this topic here.

13.3 Variable subinterval lengths

It is, of course, not necessary to keep the 1 : 2 ratio of subinterval lengths. For example, if there is a point s_0 such that $\delta_{[0,s_0],D}(f)$ is much greater than $\delta_{[s_0,T],D}(f)$, the halving, or any other kind of division, is natural to be continued on $[0, s_0]$. This reminds us of the idea used in the adaptive numerical methods with a variable step length.

13.4 Applicability on small intervals

In contrast to purely numerical approaches, where one may be forced to discretise with a tiny step, the efficiency of the technique based on Theorem 6.5 is not so much affected by the smallness of the interval. This makes the scheme well applicable, in particular, for the study of high-frequency oscillations.

13.5 Advantages over other methods

The proposed technique has some other positive features distinguishing it from other approaches. For example, when applying it, one experiences no difficulties with the selection of the starting approximation (in contrast, *e.g.*, to monotone iterative methods); there is no need to re-calculate considerable amounts of data when passing to the next step of approximation (unlike projection methods); the global Lipschitz condition and the assumption on the unique solvability of the Cauchy problem are not necessary (unlike shooting method); *etc.* As regards the last mentioned condition, one should note that, for functional differential equations, it is violated even in very simple cases, and it is thus unnatural to require it when constructing a scheme of analysis of a reasonably wide class of problems.

13.6 Repeated interval halving

The interval halving procedure can be repeated. When doing so, we observe that conditions both on the eigenvalues of the Lipschitz matrix and the size of the domain are weakened by half at each step. Indeed, it follows immediately from Corollary 6.7 that the periodic successive approximation scheme constructed with k interval halvings is applicable provided that

$$r(K) < \frac{2^k}{T\varrho_*} \tag{13.1}$$

and

$$D\left(\frac{T}{2^{k+2}}\delta_{[0,T],D}(f)\right) \neq \varnothing. \tag{13.2}$$

It is also clear that the $D(2^{-k-2}T\delta_{[0,T],D}(f))$, $k = 0, 1, \ldots$, is a strictly increasing sequence of sets tending to the original domain D in the limit as k grows to ∞. In other words, rather interestingly, the scheme suggested here is theoretically applicable however large the eigenvalues of K may be.

The side-effect of the successive interval halving is the increase of the dimension of the system of determining equations, which contains $2^k n$ equations at the kth interval halving. One can regard this as a certain price to be paid for being able to apply interval halving in order to convert a divergent iteration scheme into a convergent one.

In this way, by carrying out interval halving sequentially, one can, in particular, re-establish the convergence of numerical-analytic algorithms for systems of ordinary differential equations with globally Lipschitzian non-linearities (see [12, 25, 26]).

13.7 Combination with other methods

The most difficult part of the scheme discussed consists in the analytic construction of so many members of the parametrised iteration sequence (9.4) which is sufficient to establish the solvability of the periodic problem (see conditions (10.6), (10.7)) and achieve the required precision of approximation in (11.1). Its practical implementation, usually done by using symbolic computation systems, can be considerably facilitated by combining the analytic computation with a suitable kind of approximation. The use of the polynomial or trigonometric interpolation (see [10, 27]) is very convenient for this purpose.

13.8 Non-degeneracy condition for higher-order approximations

It is obvious from (9.7) and (10.8) that $\lim_{m\to\infty} M_m = 0$ and, hence, the right-hand side of inequality (10.7) vanishes when m grows to $+\infty$. On the other hand, it is easy to see that, under the conditions assumed, the mapping Φ_m (uniformly on compact sets) converges to Φ as m tends to $+\infty$. We thus arrive at the interesting observation that assumption (10.7) of Theorem 10.2, which is the main condition ensuring the non-degeneracy of the homotopy, has the form of the strict inequality

$$|\Phi_m| \rhd_{\partial\Omega} w_m,$$

where $|\Phi_m|$ approaches to $|\Phi|$ while the term w_m becomes arbitrarily small as m grows to $+\infty$.

13.9 Relation to continuation theorems

Theorem 10.2 and similar statements can also be applied on the zeroth step of iteration, *i.e.*, when one does not perform any iteration at all. This reminds us of the notion of a generating system appearing, *e.g.*, in the asymptotic methods.

Indeed, having in mind Theorem 10.2 in its present formulation and recalling condition (10.12), let us put

$$f^{\#}(\xi,\eta) := \begin{pmatrix} \eta - \xi - \int_0^{\frac{T}{2}} f(\tau, (1-\frac{2t}{T})\xi + \frac{2t}{T}\eta)\,d\tau \\ \xi - \eta - \int_{\frac{T}{2}}^{T} f(\tau, (\frac{2t}{T}-1)\xi + 2(1-\frac{t}{T})\eta)\,d\tau \end{pmatrix} \tag{13.3}$$

for any $(\xi,\eta) \in G_D(r_D(f))$. Recall that $G_D(r_D(r))$ is a subset of D^2 which *a priori* contains the value $(u(0), u(\frac{1}{2}T))$ for the periodic solution $u(\cdot)$ in question.

By using Theorem 10.2 for $m = 0$ with condition (10.7) replaced by (10.12), we obtain the following statement on the solvability of the periodic problem (1.3), (1.4).

Corollary 13.1 *Let assumption* (6.20) *hold and let the convergence condition* (9.7) *be satisfied. Furthermore, let there exist a closed domain* $\Omega \subset G_D(r_D(f))$ *such that*

$$\deg\big(f^{\#}, \Omega\big) \neq 0 \tag{13.4}$$

and

$$\big|f^{\#}\big| \rhd_{\partial\Omega} \frac{5T^2}{108} \begin{pmatrix} K(1_n - \frac{3}{20}TK)^{-1}\delta_{[0,\frac{1}{2}T],D}(f) \\ K(1_n - \frac{3}{20}TK)^{-1}\delta_{[\frac{1}{2}T,T],D}(f) \end{pmatrix}. \tag{13.5}$$

Then the periodic boundary value problem (1.3), (1.4) *has at least one solution* $u(\cdot)$ *which has values in* D *and, moreover, is such that* $(u(0), u(\frac{1}{2}T)) \in \Omega$.

Recall that the vectors $\delta_{[0,\frac{1}{2}T],D}(f)$ and $\delta_{[\frac{1}{2}T,T],D}(f)$ are computed directly according to formula (2.1), whereas '$\triangleright_{\partial\Omega}$' means that, at every point from $\partial\Omega$, the strict inequality '>' holds for at least one row, and the number of that row may vary with the point.

Assumptions of type (6.12), (13.5) are natural from various points of view. For example, let us imagine for a while that no interval halving has been carried out at all and thus, instead of Theorem 6.5, we are in the situation described by Proposition 3.1 with $g = f$, $p = T$ and $t_0 = 0$. The system of $2n$ determining equations (8.2) then turns back into the n-dimensional system (3.15),

$$\int_0^T f\big(t, u_\infty(t,\xi)\big)\,dt = 0,$$

the zeroth approximation of which, in the sense of the iteration process (3.4), has the form

$$\int_0^T f(t,\xi)\,dt = 0. \tag{13.6}$$

Therefore, assumption (6.12) becomes

$$\deg(\bar{f}, V) \neq 0 \tag{13.7}$$

with a suitable domain $V \subset D$, where

$$\bar{f}(\xi) := \int_0^T f(t,\xi)\,dt \tag{13.8}$$

for $\xi \in V$. Then, using [12, Lemma 3.26] with $m = 0$, one easily shows that the following statement holds.

Corollary 13.2 *The conditions* (13.7), $r(K) < 10(3T)^{-1}$ *and*

$$|\bar{f}| \triangleright_{\partial V} \frac{5T^2}{27} K\left(1_n - \frac{3}{10}TK\right)^{-1} \delta_{[0,T],D}(f) \tag{13.9}$$

are sufficient for the solvability of the periodic problem (1.3), (1.4).

Arguing in this manner, we can obtain, in particular, the well-known Mawhin's theorem [28], with (13.7) being the solvability condition for the generating equation (of course, one could use the condition of *a priori* bounds type instead of (13.9) for a more exact resemblance). In this context, Corollary 13.1 can be regarded as a 'halved' analogue of the last mentioned statement, where the equations

$$\int_0^{\frac{T}{2}} f\left(\tau, \left(1 - \frac{2t}{T}\right)\xi + \frac{2t}{T}\eta\right) d\tau = \eta - \xi, \tag{13.10}$$

$$\int_{\frac{T}{2}}^T f\left(\tau, \left(\frac{2t}{T} - 1\right)\xi + 2\left(1 - \frac{t}{T}\right)\eta\right) d\tau = \xi - \eta \tag{13.11}$$

determine the initial data of the zeroth approximation. The side-effect of halving is visible from the presence of two independent variables, ξ and η, due to which system (13.10), (13.11), in contrast to (13.6), contains n extra equations.

It should be noted that the convergence of the iteration scheme in Corollary 13.1 is guaranteed under the assumption $r(K) < 20(3T)^{-1}$, which is twice as weak as the corresponding condition of Corollary 13.2 ($r(K) < 10(3T)^{-1}$).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The initial draft was prepared mainly by the first two authors, while the third one carried out the numerical computations and an overall check of estimates. All the authors contributed equally to the final version of this work and approved its present form.

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Notes on interval halving procedure for periodic and two-point problems

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Abstract

We continue our study of constructive numerical-analytic schemes of investigation of boundary problems. We simplify and improve the recently suggested interval halving technique allowing one to essentially weaken the convergence conditions.

MSC: Primary 34B15

Keywords: periodic solution; two-point problem; Lyapunov-Schmidt reduction; determining equation; parametrisation; periodic successive approximations; numerical-analytic method; Cesari method; interval halving

1 Introduction

The present note is a continuation of [1] and deals with a constructive approach to the investigation of two-point boundary value problems. The approach is numerical-analytic [2, 3] in the sense that, although part of the computation is carried out analytically, the final stage of the method involves a numerical analysis of certain equations usually referred to as *determining*, or *bifurcation*, equations. This scheme of Lyapunov-Schmidt type [4, 5] reminds one of the shooting method on first glance, but there are several essential differences [1].

We consider the periodic boundary value problem

$$u'(t) = f\big(t, u(t)\big), \quad t \in [0, p], \tag{1}$$

$$u(0) = u(p), \tag{2}$$

where $p \in (0, \infty)$, $f : [0, p] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Carathéodory conditions, and a solution is an absolutely continuous vector function satisfying (1) almost everywhere on $[0, p]$. Our main assumption till the end of the paper is that there exist a certain matrix K and a bounded closed set $\Omega \subset \mathbb{R}^n$ such that $f(t, \cdot) \in \mathrm{Lip}_K(\Omega)$ for a.e. $t \in [0, p]$. Here and below, given a square matrix K with non-negative entries, $\mathrm{Lip}_K(\Omega)$ stands for the set of functions $g : \Omega \to \mathbb{R}^n$ satisfying the componentwise Lipschitz condition

$$\big|g(z_1) - g(z_2)\big| \le K|z_1 - z_2| \tag{3}$$

for all z_1 and z_2 from Ω. In (3) and all similar relations that will appear below, the symbols $\le$ and $|\cdot|$ are understood componentwise.

In its original form (see, *e.g.*, [2, 3] for references), the numerical-analytic approach that we are dealing with suggests one to look for a solution of (1), (2) among the limit functions of certain n-parametric family of sequences possessing property (2) (see, *e.g.*, [2, 3]). Given an arbitrary vector ξ, consider the sequence of functions defined by the recurrence relation

$$u_m(t,\xi) := \xi + \int_0^t f\big(s, u_{m-1}(s,\xi)\big)\,ds - \frac{t}{p}\int_0^p f\big(s, u_{m-1}(s,\xi)\big)\,ds, \quad t \in [0,p], \tag{4}$$

with $m = 1, 2, \ldots$ and $u_0(t,\xi) := \xi$, $t \in [0,p]$. Clearly, each of functions (4) satisfies the periodic boundary condition (2). If one establishes the existence of the limit

$$u_\infty(\cdot,\xi) := \lim_{m\to\infty} u_m(\cdot,\xi), \quad \xi \in \Omega, \tag{5}$$

with a certain $\Omega \subset D$, one finds out that the existence of a solution $u(\cdot)$ of the periodic problem (1), (2) with the value at zero lying in Ω is equivalent to the solvability of the equation

$$\int_0^p f\big(s, u_\infty(s,\xi)\big)\,ds = 0$$

with respect to the unknown vector ξ. This leads to a Lyapunov-Schmidt type reduction of the periodic problem (see [1, 3] for more details), which is known to be applicable on the assumption that

$$f(t,\cdot) \in \mathrm{Lip}_K(D) \quad \text{for a.e. } t \in [0,p], \tag{6}$$

with pK small enough and D satisfying the condition

$$D\left(\frac{T}{4}\delta_D(f)\right) \neq \varnothing, \tag{7}$$

where

$$\delta_D(f) := \max\big\{\delta_{[0,p/2],D}(f), \delta_{[p/2,p],D}(f)\big\} \tag{8}$$

and $\delta_{J,V}(f) := \max_{(t,\xi)\in J\times V} f(t,\xi) - \min_{(t,\xi)\in J\times V} f(t,\xi)$ for any compact $V \subseteq D$ and $J \subseteq [0,p]$.

In (7), $D(\varrho)$ is the *ϱ-core* of D defined as

$$D(\varrho) := \big\{z \in D : B(z,\varrho) \subset D\big\} \tag{9}$$

for any non-negative vector ϱ, where

$$B(z,\varrho) := \big\{\xi \in \mathbb{R}^n : |z - \xi| \leq \varrho\big\} \tag{10}$$

is the (componentwise) ϱ-neighbourhood of z. Note that Ω involved in (5) is actually equal to $D(\varrho)$ in that case. Examples showing how $D(\varrho)$ can look like for different D can be seen on Figure 1.

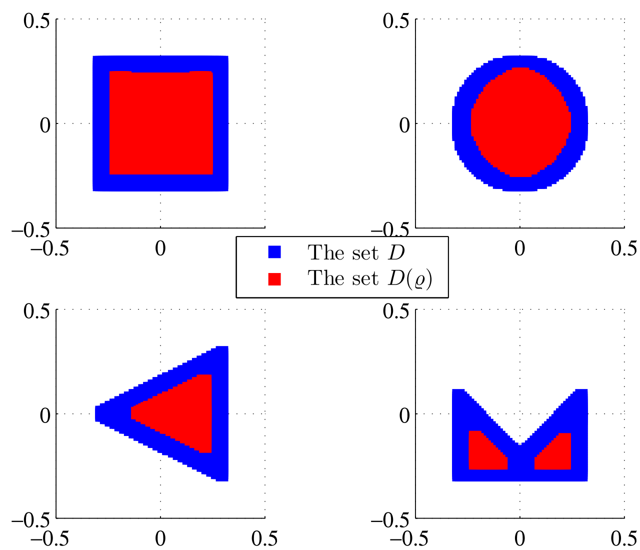

Figure 1 The set $D(\varrho)$ in examples, $\varrho = \mathrm{col}(1/12, 1/18)$.

The main limitation of this approach is that, in order to guarantee the convergence, one has to assume a certain smallness of the eigenvalues of the matrix pK. It was shown, in particular, in [6] that the method based upon sequence (4) is applicable provided that

$$r(K) < \frac{1}{\gamma_0 p}, \tag{11}$$

where

$$\gamma_0 := \frac{3}{10}. \tag{12}$$

Moreover, as is seen from condition (7), the set D where (6) holds should be wide enough (in particular, such that $\operatorname{diam} D \geq \frac{p}{2}\delta_D(f)$, with the natural componentwise definition of a vector-valued diameter of a set).

As the recent paper [1] shows, the limitation can be overcome by noticing that the quantity which is assumed be small enough is always proportional to the length of the interval. A natural interval halving technique then allows one to produce a version of the scheme where (11) is replaced by the condition

$$r(K) < \frac{2}{\gamma_0 p}$$

and, thus, weakened by half. A similar improvement is also achieved in relation to condition (7), which is replaced by the assumption that

$$D\left(\frac{T}{8}\delta_D(f)\right) \neq \varnothing. \tag{13}$$

Clearly, the transition to (13) weakens (7) by half.

Here, we modify the scheme of [1] so that its substantiation is simplified and, in particular, replace (7) by an assumption which is more transparent and, generally speaking, less restrictive. Indeed, the idea to start from a set D where the nonlinearity is known to be Lipschitzian and look for its suitable subset $D(\varrho)$ that could potentially contain initial values of periodic solutions is somewhat unnatural because, in any case, it is the initial values that are of major interest, the regularity assumptions for the equation being only technical assumptions induced by the method. Instead of doing so, which used to be the case in [1] and in all the previous works, it is, however, more logical to choose a closed bounded set $\Omega \subset \mathbb{R}^n$, where one expects to find initial values of the solution, and to assume that the nonlinearity is Lipschitzian on a suitable $\tilde{\Omega} \supset \Omega$, with $\tilde{\Omega}$ only as large as the method requires. It is not difficult to see that the argument of [1] then leads us to the choice $\tilde{\Omega} := \Omega_\varrho$, where Ω_ϱ is the ϱ-*neighbourhood* of Ω in the sense that

$$\Omega_\varrho := \bigcup_{\xi \in \Omega} B(\xi, \varrho), \tag{14}$$

where the symbol $B(\xi, \varrho)$ stands for the ϱ-neighbourhood of a vector ξ (recall that the relations in (10) are componentwise). Besides its more natural character, the use of the pair of sets (Ω_ϱ, Ω) is also advantageous in contrast to $(D, D(\varrho))$ because, geometrically, $D(\varrho)$ does not necessarily copy the shape of D (see Figures 2 and 3 for examples where D and the corresponding $D(\varrho)$ with $\varrho = \binom{\varrho_1}{\varrho_2}$ gradually increasing are represented, respectively, by the blue and red regions). In fact, the operations of taking ϱ-core and ϱ-neighbourhood do not commute: the equality in the inclusion

$$\big(D(\varrho)\big)_\varrho \subset D \tag{15}$$

is, in general, not true, whereas one obviously has

$$\Omega_\varrho(\varrho) = \Omega \tag{16}$$

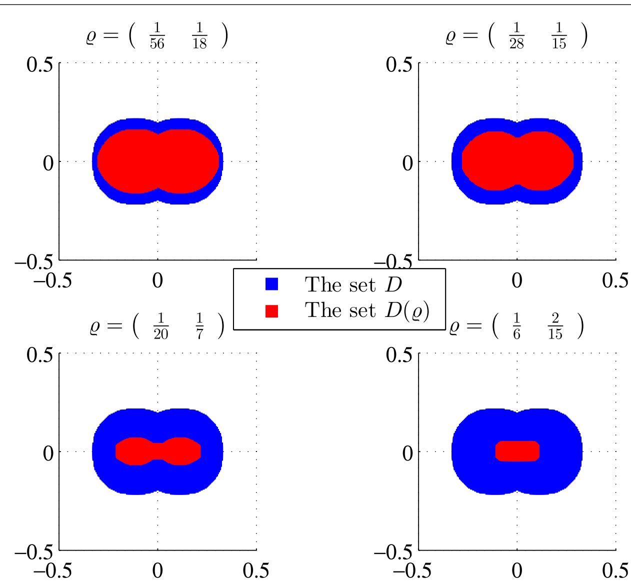

Figure 2 The shape of $D(\varrho)$ may differ significantly for various values of ϱ.

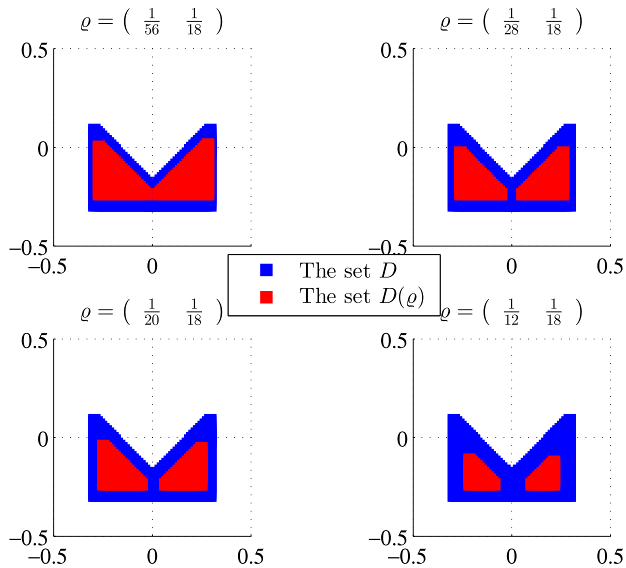

Figure 3 An example of *D* with *D*(ϱ) disconnected for certain ϱ.

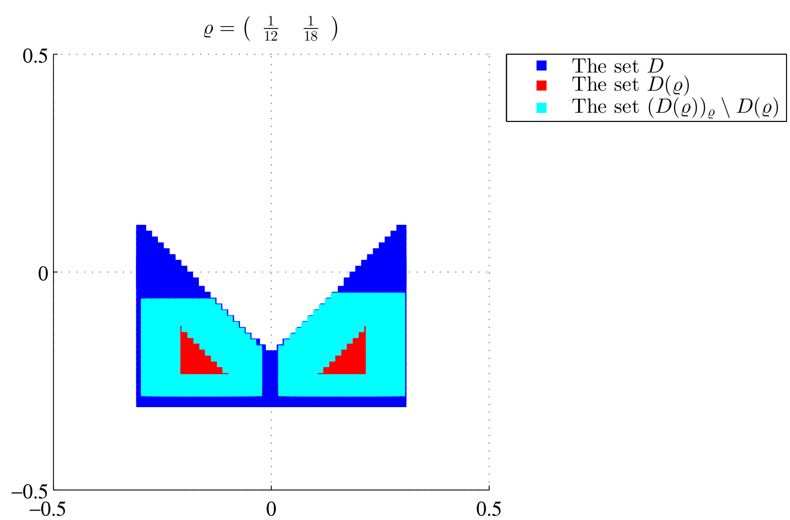

Figure 4 An example of the set *D* illustrating the strict inclusion in (15).

for any ϱ. The strict inclusion in (15) holds, in particular, in the example from Figure 4, where the points of the sets $D(\varrho)$, D and $(D(\varrho))_\varrho$ for several values of $\varrho = \binom{\varrho_1}{\varrho_2}$, $\varrho_1 \le \varrho_2$, are plotted in red, blue and cyan, respectively. In that example, by choosing Ω to be the red region, one should then widen it for the technical purposes related to the method up to the cyan one, and not the blue one. A comparison of (15) and (16) confirms the advantage of assuming conditions of type (6) on Ω_ϱ. Several examples of domains Ω and the corresponding sets Ω_ϱ can be seen on Figure 5.

Using the (Ω_ϱ, Ω) setting, we further reformulate the scheme of the method further by removing certain unnecessary technicalities so that both the setting and the overall analysis are simplified. The new formulation, in particular, makes it particularly easy to adopt

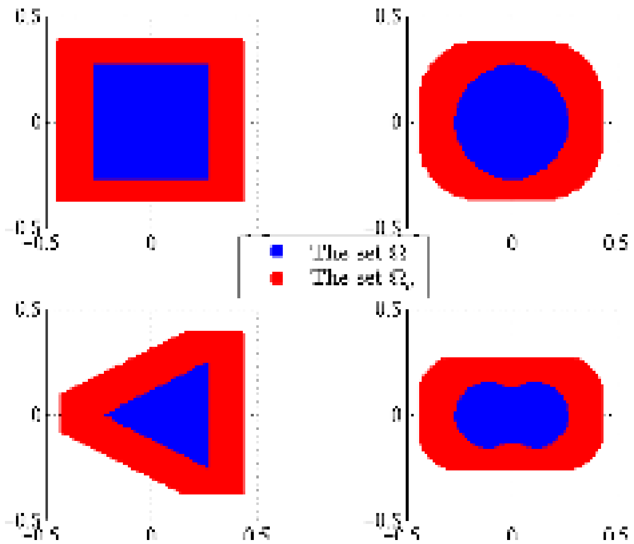

Figure 5 The set Ω_ϱ in examples, $\varrho = \mathrm{col}(1/8, 1/16)$.

the approach to problems with two-point boundary conditions different from the periodic ones, which technique is also outlined in what follows.

2 Construction of iterations and proof of convergence

Thus, let us fix a closed bounded set $\Omega \subset \mathbb{R}^n$, where the initial values of solutions of problem (1), (2) will be looked for. Without loss of generality, we shall choose Ω to be convex. Let ξ and η be arbitrary vectors from Ω. Let us put

$$x_0(t,\xi,\eta) := \left(1 - \frac{2t}{p}\right)\xi + \frac{2t}{p}\eta, \quad t \in [0,p/2], \tag{17}$$

$$y_0(t,\xi,\eta) := 2\left(1 - \frac{t}{p}\right)\eta + \left(\frac{2t}{p} - 1\right)\xi, \quad t \in [p/2,p], \tag{18}$$

and define the recurrence sequences of functions $x_m : [0,p/2] \times \Omega^2 \to \mathbb{R}^n$ and $y_m : [p/2,p] \times \Omega^2 \to \mathbb{R}^n$, $m = 0,1,\ldots$, according to the formulae

$$\begin{aligned} x_m(t,\xi,\eta) &:= x_0(t,\xi,\eta) + \int_0^t f\bigl(s, x_{m-1}(s,\xi,\eta)\bigr)\,ds \\ &\quad - \frac{2t}{p}\int_0^{\frac{p}{2}} f\bigl(s, x_{m-1}(s,\xi,\eta)\bigr)\,ds, \quad t \in [0,p/2], \end{aligned} \tag{19}$$

$$\begin{aligned} y_m(t,\xi,\eta) &:= y_0(t,\xi,\eta) + \int_{\frac{p}{2}}^t f\bigl(s, y_{m-1}(s,\xi,\eta)\bigr)\,ds \\ &\quad - \left(\frac{2t}{p} - 1\right)\int_{\frac{p}{2}}^{p} f\bigl(s, y_{m-1}(s,\xi,\eta)\bigr)\,ds, \quad t \in [p/2,p], \end{aligned} \tag{20}$$

where $m \geq 0$. One arrives at formulae (17), (18) directly when choosing $x_0(\cdot,\xi,\eta)$ and $y_0(\cdot,\xi,\eta)$ as linear functions on the appropriate intervals satisfying the equalities

$$x_0(0,\xi,\eta) = \xi, \qquad x_0\left(\frac{p}{2},\xi,\eta\right) = \eta, \tag{21}$$

$$y_0\left(\frac{p}{2},\xi,\eta\right)=\eta,\qquad y_0(p,\xi,\eta)=\xi. \tag{22}$$

The considerations in [1] concerning the auxiliary parametrised problems (4.1), (4.2) and (4.3), (4.4) can be omitted. Clearly, (17) and (18) are the simplest choice of functions satisfying (21) and (22).

The form of sequences (19), (20) is motivated by the following proposition.

Proposition 1 *Let* $(\xi,\eta)\in\Omega^2$ *be fixed. If the limits* $x_\infty(\cdot,\xi,\eta)$ *and* $y_\infty(\cdot,\xi,\eta)$ *of sequences* (19) *and* (20), *respectively, exist uniformly on* $[0,\frac{1}{2}p]$ *and* $[\frac{1}{2}p,p]$, *then*:

1. *The function* $x_\infty(\cdot,\xi,\eta)$ *has the property*

$$x_\infty\left(\frac{p}{2},\xi,\eta\right)-x_\infty(0,\xi,\eta)=\eta-\xi \tag{23}$$

and is the unique solution of the initial value problem

$$x'(t)=f\bigl(t,x(t)\bigr)+\frac{2}{p}\Xi(\xi,\eta),\quad t\in[0,p/2], \tag{24}$$

$$x(0)=\xi, \tag{25}$$

where

$$\Xi(\xi,\eta):=\eta-\xi-\int_0^{\frac{p}{2}}f\bigl(\tau,x_\infty(\tau,\xi,\eta)\bigr)\,d\tau. \tag{26}$$

2. *The function* $y_\infty(\cdot,\xi,\eta)$ *has the property*

$$y_\infty(p,\xi,\eta)-y_\infty\left(\frac{p}{2},\xi,\eta\right)=\xi-\eta \tag{27}$$

and is the unique solution of the initial value problem

$$y'(t)=f\bigl(t,y(t)\bigr)+\frac{2}{p}\mathrm{H}(\xi,\eta),\quad t\in[p/2,p], \tag{28}$$

$$y\left(\frac{p}{2}\right)=\eta, \tag{29}$$

where

$$\mathrm{H}(\xi,\eta):=\xi-\eta-\int_{\frac{p}{2}}^{p}f\bigl(\tau,y_\infty(\tau,\xi,\eta)\bigr)\,d\tau. \tag{30}$$

The proposition stated above, which is an easy consequence of the definitions of the functions $x_m:[0,p/2]\times\Omega^2\to\mathbb{R}^n$ and $y_m:[p/2,p]\times\Omega^2\to\mathbb{R}^n$, $m=0,1,\dots$, suggests one to consider the function $u_\infty(\cdot,\xi,\eta):[0,p]\to\mathbb{R}^n$ introduced according to the formula

$$u_\infty(t,\xi,\eta):=\begin{cases}x_\infty(t,\xi,\eta) & \text{if } t\in[0,p/2],\\ y_\infty(t,\xi,\eta) & \text{if } t\in(p/2,p]\end{cases} \tag{31}$$

for all ξ and η from Ω and look for solutions of problem (1), (2) in the form $u_\infty(\cdot,\xi,\eta)$. Note that, as follows immediately from (23), (25) and (29),

$$x_\infty\left(\frac{p}{2},\xi,\eta\right) = y_\infty\left(\frac{p}{2},\xi,\eta\right)$$

and, therefore, the function $u_\infty(\cdot,\xi,\eta)$ is continuous on $[0,p]$ for any $(\xi,\eta) \in \Omega^2$. The use of this function, however, requires the knowledge of the fact that $x_\infty(\cdot,\xi,\eta)$ and $y_\infty(\cdot,\xi,\eta)$ are well defined for $(\xi,\eta) \in \Omega^2$.

Introduce the functions

$$\bar{\alpha}_1(t) := \frac{2}{p}t(p-t), \quad t \in [0,p/2] \tag{32}$$

and

$$\bar{\bar{\alpha}}_1(t) := \frac{1}{2p}(p-2t)(2t-3p), \quad t \in [p/2,p]. \tag{33}$$

Theorem 2 *If there exists a non-negative vector* ϱ *with the property*

$$\varrho \geq \frac{p}{8}\delta_{\Omega_\varrho}(f) \tag{34}$$

such that $f(t,\cdot) \in \mathrm{Lip}_K(\Omega_\varrho)$ *for a.e.* $t \in [0,p]$ *with a certain* K *and*

$$r(K) < \frac{2}{\gamma_0 p} \tag{35}$$

then, for all fixed $(\xi,\eta) \in \Omega^2$*, the sequence* $\{x_m(\cdot,\xi,\eta) : m \geq 0\}$ *(resp.,* $\{y_m(\cdot,\xi,\eta) : m \geq 0\}$*) converges to a limit function* $x_\infty(\cdot,\xi,\eta)$ *(resp.,* $y_\infty(\cdot,\xi,\eta)$*) uniformly in* $t \in [0,p/2]$ *(resp.,* $t \in [p/2,p]$*), and the following estimates hold*:

$$\left|x_m(\cdot,\xi,\eta) - x_\infty(t,\xi,\eta)\right| \leq \frac{\bar{\alpha}_1(t)}{2^{m+1}}(\gamma_0 pK)^m\left(\mathbf{1}_n - \frac{\gamma_0 p}{2}K\right)^{-1}\delta_{[0,p/2],\Omega_\varrho}(f) \tag{36}$$

for all $t \in [0,p/2]$ *and*

$$\left|y_m(\cdot,\xi,\eta) - y_\infty(t,\xi,\eta)\right| \leq \frac{\bar{\bar{\alpha}}_1(t)}{2^{m+1}}(\gamma_0 pK)^m\left(\mathbf{1}_n - \frac{\gamma_0 p}{2}K\right)^{-1}\delta_{[p/2,p],\Omega_\varrho}(f) \tag{37}$$

for all $t \in [p/2,p]$ *and* $m \geq 3$.

In estimates (36) and (37), the symbol $\mathbf{1}_n$ stands for the unit matrix of dimension n. Recall also that $\gamma_0 = 3/10$, as indicated above. Note that condition (35) can be slightly improved by replacing γ_0 by the constant

$$\gamma_* \approx 0.2927 \tag{38}$$

(see [1, 3] for more details). The unpleasant side effect is, however, that estimates (36) and (37) under such a condition are established for m sufficiently large only, which puts an obstacle in obtaining efficient solvability conditions in Corollary 8 below. This circumstance is not actually of primary importance since the very aim of the interval halving technique discussed here is to weaken assumption (35) by half and, in any case, the difference between the two conditions is quite insignificant because $\gamma_0 - \gamma_* \approx 0.00727$.

Remark 3 It follows from [3, Lemma 3.12] that estimates (36) and (37) can be shown to hold for all $m \geq 0$ if the definition of functions $\bar{\alpha}_1$ and $\bar{\bar{\alpha}}_1$ is changed slightly (namely, the multiplier 10/9 is added on the right-hand side of (32), (33)).

It should be mentioned that assumption (35), which, by Theorem 2, ensures the applicability of the iteration scheme based on formulae (19), (20), is twice as weak as assumption (11) for the original sequence (4). The same kind of improvement is achieved concerning the condition on the set D where f is Lipschitzian since, for the scheme without interval halving, one would require that

$$\exists \varrho: \quad \varrho \geq \frac{p}{4}\delta_{\Omega_\varrho}(f), \tag{39}$$

which is twice as strong as (34). In contrast to the related assumptions from [1] and earlier works, condition (34) is easier to verify because in order to do so one has only to find the value $\delta_{\Omega_\varrho}(f)$, which is computed directly by estimating f. In addition, it is possible to estimate this value in certain cases where some further information on the behaviour of f is known.

Comparing Theorem 2 with Theorems 6.1 and 6.3 of [1], where the values in (6.5) and (6.12) are computed over the entire domain where f is Lipschitzian, we see that the values $\delta_{[0,p/2],\Omega_\varrho}(f)$ and $\delta_{[p/2,p],\Omega_\varrho}(f)$ in Theorem 2 are computed over Ω_ϱ only.

The proof of Theorem 2 is carried out by a suitable modification of that of [1, Theorem 6.5] and is based upon the following lemmata.

Lemma 4 ([1, Lemma 7.1]) *Let* $x : [0,p/2] \to \mathbb{R}^n$ *and* $y : [p/2,p] \to \mathbb{R}^n$ *be arbitrary functions such that* $\{x(t) : t \in [0,p/2]\} \subset \Omega$ *and* $\{y(t) : t \in [p/2,p]\} \subset \Omega$. *Then*

$$\begin{aligned} \left|P_0 f\left(\cdot, x(\cdot)\right)\right|(t) &\leq \frac{1}{2}\bar{\alpha}_1(t)\delta_{[0,p/2],\Omega}(f) \\ &\leq \frac{p}{8}\delta_{[0,p/2],\Omega}(f) \end{aligned} \tag{40}$$

for $t \in [0,p/2]$ *and*

$$\begin{aligned} \left|P_1 f\left(\cdot, y(\cdot)\right)\right|(t) &\leq \frac{1}{2}\bar{\bar{\alpha}}_1(t)\delta_{[p/2,p],\Omega}(f) \\ &\leq \frac{p}{8}\delta_{[p/2,p],\Omega}(f) \end{aligned} \tag{41}$$

for $t \in [p/2,p]$.

In (40), (41), the mappings P_0 and P_1 are defined by the equality

$$(P_i v)(t) := \int_{\frac{i}{2}p}^{t} v(s)\,ds - \left(\frac{2t}{p} - i\right)\int_{\frac{i}{2}p}^{\frac{i+1}{2}p} v(s)\,ds \tag{42}$$

for all $t \in [\frac{i}{2}p, \frac{1}{2}(i+1)p]$, $i \in \{0,1\}$, and $v \in C([\frac{i}{2}p, \frac{1}{2}(i+1)p], \mathbb{R}^n)$.

Lemma 5 *Let ϱ be a vector satisfying relation* (34). *Then, for arbitrary $m \geq 0$ and $(\xi, \eta) \in \Omega^2$, the inclusions*

$$\{x_m(t, \xi, \eta) : t \in [0, p/2]\} \subset \Omega_\varrho \tag{43}$$

and

$$\{y_m(t, \xi, \eta) : t \in [p/2, p]\} \subset \Omega_\varrho \tag{44}$$

hold.

Proof Let ξ and η be arbitrary vectors from Ω. It is natural to argue by induction. Since Ω is assumed to be convex, it follows from (17) that $x_0(t, \xi, \eta) \in \Omega$ for any $t \in [0, p/2]$ and $y_0(t, \xi, \eta) \in \Omega$ for any $t \in [p/2, p]$, *i.e.*, (43) and (44) are true for $m = 0$. Let us assume that (43) and (44) hold for a certain $m = m_0$.

Considering relations (17), (19), (8), (34) and (42) and using Lemma 4, we obtain

$$\begin{aligned} \left|x_{m_0+1}(t, \xi, \eta) - x_0(t, \xi, \eta)\right| &= \left|P_0 f\left(\cdot, x_{m_0}(\cdot, \xi, \eta)\right)\right|(t) \\ &\leq \frac{p}{8} \delta_{[0,p/2],\Omega}(f) \\ &\leq \varrho \end{aligned} \tag{45}$$

for $t \in [0, p/2]$ and

$$\begin{aligned} \left|y_{m_0+1}(t, \xi, \eta) - y_0(t, \xi, \eta)\right| &= \left|P_1 f\left(\cdot, y_{m_0}(\cdot, \xi, \eta)\right)\right|(t) \\ &\leq \frac{p}{8} \delta_{[p/2,p],\Omega}(f) \\ &\leq \varrho \end{aligned} \tag{46}$$

for $t \in [p/2, p]$. Since (43) and (44) are satisfied for $m = 0$, we see from (45), (46) that all the values of $x_{m_0+1}(\cdot, \xi, \eta)$ and $y_{m_0+1}(\cdot, \xi, \eta)$ are contained in a ϱ-neighbourhood of a point from Ω, which means that (43) and (44) hold for $m = m_0 + 1$. It now remains to use the arbitrariness of m_0. □

The assertion of Theorem 2 is now obtained by replacing [1, Lemma 7.2] by Lemma 5 and arguing by analogy to the proof of Theorems 6.1 and 6.3 from [1]. Furthermore, similarly to [1], using Proposition 1 and Theorem 2, one arrives at the following.

Theorem 6 *Assume that $f(t, \cdot) \in \mathrm{Lip}_K(\Omega_\varrho)$ for a.e. $t \in [0, p]$, where ϱ is a vector with property* (34) *and K satisfies condition* (35). *Then, for every solution $u(\cdot)$ of problem* (1), (2) *with the property*

$$\{u(t) \mid t \in [0, p]\} \subset \Omega_\varrho \quad \text{and} \quad \left\{u(0), u\left(\frac{p}{2}\right)\right\} \subset \Omega, \tag{47}$$

there exists a pair (ξ_0, η_0) in Ω^2 such that $u(\cdot) = u_\infty(\cdot, \xi_0, \eta_0)$. On the other hand, the function $u_\infty(\cdot, \xi, \eta)$ is a solution of the periodic boundary value problem (1), (2) *if and only if the pair*

(ξ, η) *satisfies the system of* $2n$ *equations*

$$\begin{aligned} &\Xi(\xi, \eta) = 0, \\ &\mathrm{H}(\xi, \eta) = 0. \end{aligned} \tag{48}$$

Recall that the functions $\Xi : \Omega^2 \to \mathbb{R}^n$ and $\mathrm{H} : \Omega^2 \to \mathbb{R}^n$ are defined according to equalities (26) and (30), and the latter equalities make sense in view of Theorem 2.

3 Constructive solvability analysis

Theorem 6 provides one a formal reduction of the periodic problem (1), (2) to the system of $2n$ numerical equations (48) in the sense that the initial data $(u(0), u(p/2))$ of any solution of (1), (2) with properties (47) can be found from (48). Thus, under the conditions assumed, the question on solutions of the periodic boundary value problem (1), (2) can be replaced that of the system of numerical equations (48). A combination of Proposition 11 and Theorems 2, 6 then suggests one a scheme of investigation of the periodic boundary value problem (1), (2). The practical realisation of the scheme is based upon the so-called approximate determining functions

$$\Xi_m(\xi, \eta) := \eta - \xi - \int_0^{\frac{p}{2}} f\big(\tau, x_m(\tau, \xi, \eta)\big)\, d\tau, \tag{49}$$

$$\mathrm{H}_m(\xi, \eta) := \xi - \eta - \int_{\frac{p}{2}}^{p} f\big(\tau, y_m(\tau, \xi, \eta)\big)\, d\tau, \tag{50}$$

considered for a fixed value of m and, thus, computable explicitly. Then, as in [1], the function

$$u_m(t, \xi, \eta) := \begin{cases} x_m(t, \xi, \eta) & \text{if } t \in [0, p/2], \\ y_m(t, \xi, \eta) & \text{if } t \in (p/2, p], \end{cases} \tag{51}$$

can be used to obtain the mth approximation to a solution of problem (1), (2) provided that we are able to find certain ξ and η satisfying the mth approximate determining equations

$$\begin{aligned} &\Xi_m(\xi, \eta) = 0, \\ &\mathrm{H}_m(\xi, \eta) = 0. \end{aligned} \tag{52}$$

Furthermore, it turns out that, under natural conditions, the solvability of the periodic problem (1), (2) can be derived from that of system (52). More precisely, putting

$$\Phi_m(\xi, \eta) := \begin{pmatrix} \eta - \xi - \frac{1}{2}\int_0^p f(\frac{p-\tau}{2}, x_m(\frac{p-\tau}{2}, \xi, \eta))\, d\tau \\ \xi - \eta - \frac{1}{2}\int_0^p f(\frac{p+\tau}{2}, y_m(\frac{p+\tau}{2}, \xi, \eta))\, d\tau \end{pmatrix} \tag{53}$$

and

$$\Phi_\infty(\xi, \eta) := \begin{pmatrix} \eta - \xi - \frac{1}{2}\int_0^p f(\frac{p-\tau}{2}, x_\infty(\frac{p-\tau}{2}, \xi, \eta))\, d\tau \\ \xi - \eta - \frac{1}{2}\int_0^p f(\frac{p+\tau}{2}, y_\infty(\frac{p+\tau}{2}, \xi, \eta))\, d\tau \end{pmatrix} \tag{54}$$

for any $(\xi, \eta) \in \Omega^2$, we can state the following.

Theorem 7 *Let* $f(t,\cdot) \in \mathrm{Lip}_K(\Omega_\varrho)$ *for a.e.* $t \in [0,p]$, *where* ϱ *is a vector with property* (34) *and* K *satisfies condition* (35). *Moreover, assume that* Φ_m *satisfies the condition*

$$\deg(\Phi_m, \Omega) \neq 0 \tag{55}$$

for a certain fixed $m \geq 0$ *and there exists a continuous mapping* $Q : [0,1] \times \Omega^2$ *which does not vanish on* $(0,1) \times \partial\Omega^2$ *and is such that* $Q(0,\cdot) = \Phi_m$, $Q(1,\cdot) = \Phi_\infty$. *Then there exists a pair* $(\xi^*, \eta^*) \in \Omega^2$ *such that the function* $u := u_\infty(\cdot, \xi^*, \eta^*)$ *is a solution of the periodic boundary value problem* (1), (2) *possessing properties* (47).

It should be noted that the vector field Φ_m is finite-dimensional and, thus, the degree involved in (55) is the Brower degree.

Proof We can rely on the argument from the proof of [1, Theorem 6.5]. Indeed, a certain computation based on (53) shows that

$$\Phi_m(\xi,\eta) = \begin{pmatrix} \Xi_m(\xi,\eta) \\ \mathrm{H}_m(\xi,\eta) \end{pmatrix} \tag{56}$$

for all $(\xi,\eta) \in \Omega^2$ and, thus, (52) is necessary and sufficient for (ξ,η) to be a singular point of Φ_m. Similarly to [1], the assumptions of the theorem then allow one to construct a non-degenerate deformation of Φ_m into the vector field

$$(\xi,\eta) \ni \Omega^2 \mapsto \begin{pmatrix} \Xi(\xi,\eta) \\ \mathrm{H}(\xi,\eta) \end{pmatrix}$$

the singular points of which determine solutions of problem (1), (2) satisfying condition (47), and use the homotopy invariance of the degree. The remaining property in (47) is a consequence of Lemma 5. □

Let the binary relation $\rhd_S$ be defined [3] for any $S \subset \mathbb{R}^{2n}$ as follows: functions $g = (g_i)_{i=1}^{2n} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $h = (h_i)_{i=1}^{2n} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ are said to satisfy the relation $g \rhd_S h$ if and only if there exists a function $\nu : S \to \{1,2,\ldots,2n\}$ such that $g_{\nu(z)}(z) > h_{\nu(z)}(z)$ at every point $z \in S$. Using this relation, one can formulate an efficient condition sufficient for the solvability of problem (1), (2).

Corollary 8 *Let* $f(t,\cdot) \in \mathrm{Lip}_K(\Omega_\varrho)$ *for a.e.* $t \in [0,p]$, *where* ϱ *satisfies inequality* (34) *and* K *has property* (35). *Let, moreover,*

$$|\Phi_m| \rhd_{\partial\Omega} \frac{5p}{18} \begin{pmatrix} M_m \delta_{[0,p/2],\Omega}(f) \\ M_m \delta_{[p/2,p],\Omega}(f) \end{pmatrix} \tag{57}$$

for a certain fixed $m \geq 2$, *where*

$$M_m := \left(\frac{\gamma_0 p}{2}\right)^{m+1} K^{m+1} \left(\mathbf{1}_n - \frac{\gamma_0 p}{2} K\right)^{-1}. \tag{58}$$

Then there exists a pair $(\xi^*, \eta^*) \in \Omega^2$ *such that* $u := u_\infty(\cdot, \xi^*, \eta^*)$ *is a solution of problem* (1), (2) *possessing properties* (47).

Proof It is sufficient to apply Theorem 7 with the linear homotopy

$$Q(\theta,\xi,\eta) := (1-\theta)\Phi_m(\xi,\eta) + \theta\Phi_\infty(\xi,\eta) \tag{59}$$

for $(\xi,\eta) \in \Omega^2$, $\theta \in [0,1]$, and use estimate (10.11) from [1]. □

Recall that γ_0 in (58) is given by (12). It is important to emphasise that conditions of Corollary 8 are assumed for a *fixed m*, and all the values depending on it are evaluated in finitely many steps.

The next assertion is interesting especially because it is, in fact, based upon properties of the starting approximation and, thus, shows how a useful information can be obtained when no iterations have been carried out at all. Note that the zeroth approximation is very rough indeed in any case: the periodic solution is approximated by a piecewise linear function (see Figure 6).

With the given function f involved in (1), we associate the function $f^\# : \Omega^2 \to \mathbb{R}^{2n}$ by putting

$$f^\#(\xi,\eta) := \begin{pmatrix} \eta - \xi - \frac{1}{2}\int_0^p f(\frac{p-\tau}{2}, \frac{\tau}{p}\xi + (1-\frac{\tau}{p})\eta)\,d\tau \\ \xi - \eta - \frac{1}{2}\int_0^p f(\frac{p+\tau}{2}, \frac{\tau}{p}\xi + (1-\frac{\tau}{p})\eta)\,d\tau \end{pmatrix} \tag{60}$$

for any $(\xi,\eta) \in \Omega^2$. Note that, unlike f, the function $f^\#$ depends on the phase variables only.

Corollary 9 *Assume that there is a* ϱ *with property* (47) *and* $f(t,\cdot) \in \operatorname{Lip}_K(\Omega_\varrho)$, $t \in [0,p]$, *with K satisfying inequality* (35). *Let, furthermore,*

$$\deg(f^\#,\Omega) \neq 0 \tag{61}$$

and

$$|f^\#| \rhd_{\partial\Omega} \frac{5p^2}{108} \begin{pmatrix} K(\mathbf{1}_n - \frac{\gamma_0 p}{2}K)^{-1}\delta_{[0,\frac{1}{2}p],D}(f) \\ K(\mathbf{1}_n - \frac{\gamma_0 p}{2}K)^{-1}\delta_{[\frac{1}{2}p,p],D}(f) \end{pmatrix}. \tag{62}$$

Then the p-periodic problem (1), (2) *has at least one solution* $u(\cdot)$ *which possesses properties* (47).

Proof Equalities (17), (18), (53) and (60) imply that $f^\# = \Phi_0$. It is also easy to verify by computation that condition (62) can be rewritten in the form

$$|\Phi_0| \rhd_{\partial\Omega} \frac{5p}{18} \begin{pmatrix} \tilde{M}_0\delta_{[0,p/2],\Omega}(f) \\ \tilde{M}_0\delta_{[p/2,p],\Omega}(f) \end{pmatrix}, \tag{63}$$

where $\tilde{M}_0 := (10/9)M_0$ and M_0 is given by (58). Arguing similarly to [1, Lemma 9.2] and Corollary 8 and taking Remark 3 into account, one can show that (63) ensures the non-degeneracy of homotopy (59). The required conclusion then follows from Theorem 7. □

4 Discussion

Theorems of the kind specified above allow one to study the periodic problem (1), (2) following the lines of [1, 3]. This analysis is constructive in the sense that the assumptions can be verified efficiently and the results of computation, regarded at first only as candidates for approximate solutions, simultaneously open a way to prove the solvability in a rigorous manner. As regards the computation of iterations themselves, it is helpful to apply suitable simplified versions of the algorithm, not discussed here, which are better adopted for use with computer algebra systems. The use of polynomial approximations under similar circumstances was considered, in particular, in [7].

It is interesting to note that $f^{\#}$ involved in Corollary 9 can be considered as a 'halved' analogue of the averaged map

$$\bar{f}(\xi) := \int_0^p f(s,\xi)\,ds \tag{64}$$

for $x \in \Omega$, which arises similarly to $f^{\#}$ in the situation where no interval halving is carried out. In the latter case, one has the following statement, which is a reformulation of [1, Corollary 13.2].

Corollary 10 *Let there exist some* ϱ *with property* (34). *Let*

$$\deg(\bar{f},\Omega) \neq 0 \tag{65}$$

and $f(t,\cdot) \in \mathrm{Lip}_K(\Omega_\varrho)$, $t \in [0,p]$, *with* K *satisfying inequality* (35). *If*

$$|\bar{f}| \rhd_{\partial\Omega} \frac{5p^2}{27} K(\mathbf{1}_n - \gamma_0 pK)^{-1} \delta_{\Omega_\varrho}(f), \tag{66}$$

then the p-periodic problem (1), (2) *has a solution* $u(\cdot)$ *with properties* (47).

Assumption (65) with $\bar{f}$ given by (64) arises frequently in topological continuation theorems where the homotopy to the averaged equation is considered (see, *e.g.*, [8, 9]).

It should also be noted that, as a natural extension of the above said, one can consider a scheme with multiple interval divisions. Although the addition of intermediate nodes increases the number of equations to be solved numerically (at k_0 interval halvings, one ultimately arrives a system of 2^{k_0} determining equation with respect to 2^{k_0} variables), the important gain is the ability to apply the method regardless of the value of the Lipschitz constant.

The construction of such a scheme is based on the appropriate modification of the initial approximation, which will then depend on more parameters. Consider, *e.g.*, the transition from $k_0 = 1$ to $k_0 = 2$. Renaming the variables as $\xi = (\xi_{-1}, \xi_1)$ in the former case for more convenience and denoting the initial approximation by $u_0(\cdot,\xi)$, we rewrite (21), (22) in the form

$$u_0(0,\xi) = \xi_{-1}, \qquad u_0\left(\frac{p}{2},\xi\right) = \xi_1, \qquad u_0(p,\xi) = \xi_{-1}. \tag{67}$$

Thus, the initial approximation $u_0(\cdot,\xi)$ in the corresponding iteration scheme with one division is the linear function joining the points $(0,\xi_{-1})$, $(\frac{p}{2},\xi_1)$ and (p,ξ_{-1}). Extending this tree graph like notation to the case of two interval halvings ($k_0 = 2$) and arguing similarly, we arrive at the following equalities determining $u_0(\cdot, \xi)$:

$$\begin{aligned} &u_0(0,\xi) = \xi_{-1,-1}, \qquad u_0\left(\frac{p}{4},\xi\right) = \xi_{-1,1}, \qquad u_0\left(\frac{p}{2},\xi\right) = \xi_{1,-1}, \\ &u_0\left(\frac{3p}{4},\xi\right) = \xi_{1,1}, \qquad u_0(p,\xi) = \xi_{-1,-1}. \end{aligned} \tag{68}$$

In other words, relations (68) mean that the function $u_0(\cdot,\xi)$ for $k_0 = 2$ depends on the array of parameters $\xi = (\xi_{-1,-1}, \xi_{-1,1}, \xi_{1,-1}, \xi_{1,1})$ and is obtained by the linear interpolation of the points $(0, \xi_{-1,-1})$, $(\frac{p}{4}, \xi_{-1,1})$, $(\frac{p}{2}, \xi_{1,-1})$, $(\frac{p}{2}, \xi_{1,-1})$, $(\frac{3p}{4}, \xi_{1,1})$ and $(p, \xi_{-1,-1})$. For $k_0 \geq 3$, the structure of $u_0(\cdot,\xi)$ is completely analogous, the idea is clear from Table 1 and Figure 6: one simply draws a broken line joining the corresponding nodes. Once $u_0(\cdot,\xi)$ is constructed, the formulae for the subsequent approximations are derived automatically by rescaling the projection map to the corresponding subintervals (we do not need the corresponding explicit formulae here and, therefore, omit the details).

This observation leads one to the following algorithm of investigation of the periodic problem (1), (2):

Table 1 Variables involved in the determining equations for the respective number of interval halvings

k_0	Variables in the determining equations
0	ξ
1	ξ_{-1}, ξ_1
2	$\xi_{-1,-1}, \xi_{-1,1}, \xi_{1,-1}, \xi_{1,1}$
3	$\xi_{-1,-1,-1}, \xi_{-1,-1,1}, \xi_{-1,1,-1}, \xi_{-1,1,1}, \xi_{1,-1,-1}, \xi_{1,-1,1}, \xi_{1,1,-1}, \xi_{1,1,1}$
...	...

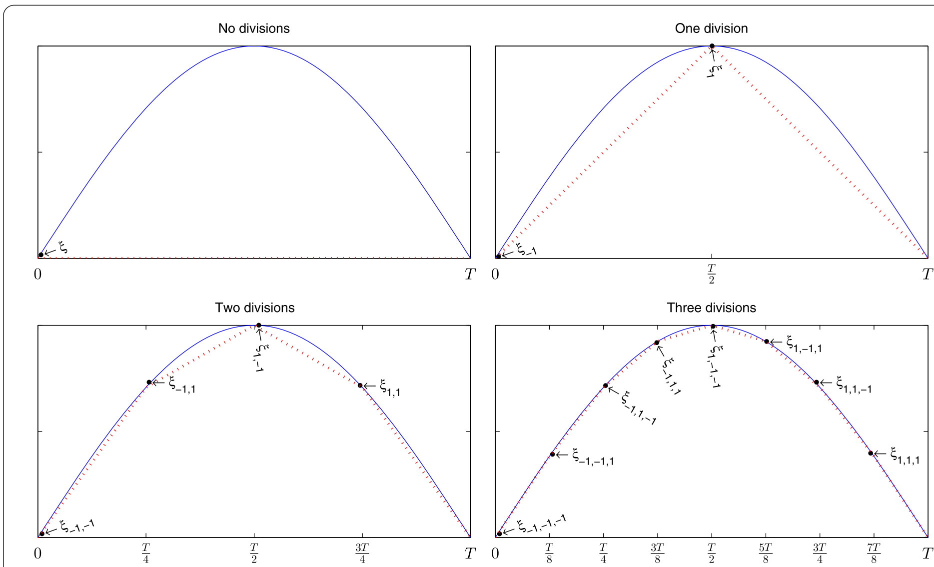

Figure 6 The meaning of parameters for the scheme with several interval divisions. In order to construct the starting approximation with any number of interval halvings, one just draws the broken line joining the corresponding nodes.

1. Fix a certain k_0 and consider the scheme with k_0 interval divisions. Fix an m_0 and construct $u_m(\cdot,\xi)$ for $m = 0,1,\dots,m_0$.
2. Solve the mth approximate determining equations for ξ, find a root $\xi^{[m]}$, and put

$$U_m(t) := u_m\big(t,\xi^{[m]}\big), \quad t \in [0,p], m = 0,1,\dots,m_0. \tag{69}$$

   In the case the equation has multiple roots, (69) and the related analysis are repeated for each of them (one can study multiple solutions of the original problem in this way).
3. 'Check' the behaviour of the functions $U_0, U_1,\dots,U_{m_0}$ (the heuristic step). If promising (*i.e.*, there are some signs of convergence), choose a suitable Ω containing the graph of U_{m_0}, find a ϱ from the condition

$$\varrho \geq \frac{p}{2^{k_0+2}}\delta_{\Omega_\varrho}(f), \tag{70}$$

   compute the Lipschitz matrix K for f in Ω_ϱ, and verify the convergence condition

$$r(K) < \frac{2^{k_0}}{\gamma_0 p}. \tag{71}$$

   If not successful with either (70) or (71), increase k_0 appropriately and try again.
4. Verify conditions of the existence theorem for Ω and m_0. If not satisfied, or if the precision of U_{m_0} is insufficient, pass to $m = m_0 + 1$ and study U_{m_0+1}. Otherwise the algorithm stops, and the outcome is:
   (a) there is a solution u of (1), (2) and $u \approx U_{m_0}$;
   (b) $\exists(\xi_*, \eta_*) \in \Omega^2$: $u(\cdot) = u_\infty(\cdot,\xi_*,\eta_*)$;
   (c) the space localisation of the graph of u is described by properties (47).

Note the role of interval divisions in the algorithm: for K not satisfying the smallness condition (11) and $k_0 = 0$ (*i.e.*, when u_m is constructed according to (4) without any interval divisions), the algorithm would stop at step 3 without any result. However, it is obvious that (70) and (71) are both satisfied if k_0 is chosen to be large enough.

In relation to the last remark, it is interesting to compare the approach discussed here with the Cesari method [10], which likewise provides one a way to reduce the periodic problem (1), (2) to a system of finitely many numerical equations. The idea of construction of the iterations there is based, in the notation of [11], on the use of the operator

$$H_m u := L - P_m L \tag{72}$$

in a suitable space of p-periodic functions, where

$$(Ly)(t) := \int_0^t y(s)\,ds - \frac{t}{p}\int_0^p y(s)\,ds, \quad t \in [0,p],$$

m is fixed, and P_m stands for the mth partial sum of the Fourier series of the corresponding function. There are visible similarities between the two approaches and, most importantly, the scheme of Cesari is also proved to be applicable regardless on the smallness of the Lipschitz constant (see [11]). The number of resulting determining equations therewith depends on the Lipschitz constant of f as well (in fact, it grows with m, the convergence being guaranteed by suitable properties of H_m for m large enough), which reminds us of Table 1 in our case. The approach presented in this note, in our opinion, has the advantage that, firstly, the computation of iterations is significantly simpler (apart of the integral mean, one does not need to compute any higher order terms in the Fourier expansion) and, secondly, it can be used for other problems as well, whereas, due to the nature of formula (72), the use of Cesari's scheme is limited to periodic functions.

In particular, the method described above is rather easy to adopt for application to two-point boundary value problems different from the periodic ones. Indeed, consider the problem with linear two-point conditions where one of the coefficient matrices is non-singular. Without loss of generality, we can assume that the problem has the form

$$u'(t) = f\big(t, u(t)\big), \quad t \in [0, p], \tag{73}$$

$$u(p) - Au(0) = c, \tag{74}$$

where A is a square matrix of dimension n (possibly, singular), $c \in \mathbb{R}^n$, $f : [0, p] \times \mathbb{R}^n \to \mathbb{R}^n$, and $p \in (0, \infty)$.

The transition from the periodic problem (1), (2) to problem (73), (74) is then surprisingly simple: one does not need but to adjust the functions $x_0(\cdot, \xi, \eta)$ and $y_0(\cdot, \xi, \eta)$ so that they satisfy the boundary condition (74). More precisely, let us fix a suitable Ω and take arbitrary ξ and η in it. Introduce the sequences of functions $x_m : [0, p/2] \times \Omega^2 \to \mathbb{R}^n$ and $y_m : [p/2, p] \times \Omega^2 \to \mathbb{R}^n$, $m = 0, 1, \ldots$, according to the same recurrence formulae as in (19), (20), where, instead of (17), (18), the functions $x_0(\cdot, \xi, \eta)$ and $y_0(\cdot, \xi, \eta)$ are given by the equalities

$$x_0(t, \xi, \eta) := \left(1 - \frac{2t}{p}\right)\xi + \frac{2t}{p}\eta, \quad t \in [0, p/2], \tag{75}$$

$$y_0(t, \xi, \eta) := 2\left(1 - \frac{t}{p}\right)\eta + \left(\frac{2t}{p} - 1\right)(A\xi + c), \quad t \in [p/2, p]. \tag{76}$$

Clearly, $x_0(\cdot, \xi, \eta)$ and $y_0(\cdot, \xi, \eta)$ given by (75), (76) are the linear functions satisfying the equalities

$$x_0(0, \xi, \eta) = \xi, \qquad x_0\left(\frac{p}{2}, \xi, \eta\right) = \eta, \tag{77}$$

$$y_0\left(\frac{p}{2}, \xi, \eta\right) = \eta, \qquad y_0(p, \xi, \eta) = A\xi + c, \tag{78}$$

which reduce to (17), (18) if A is the unit matrix and $c = 0$. Then, similarly to Proposition 1, it is not difficult to prove the following.

Proposition 11 *Let* $(\xi, \eta) \in \Omega^2$ *be fixed. If the limits* $x_\infty(\cdot, \xi, \eta)$ *and* $y_\infty(t, \xi, \eta)$ *of sequences* (19) *and* (20) *exist, then*:

1. *The function* $x_\infty(\cdot, \xi, \eta)$ *has the property*

$$x_\infty\left(\frac{p}{2}, \xi, \eta\right) - Ax_\infty(0, \xi, \eta) = \eta - A\xi \tag{79}$$

and is the unique solution of the initial value problem (24), (25) *with* Ξ *given by* (26).

2. *The function* $y_\infty(\cdot,\xi,\eta)$ *has the property*

$$y_\infty(p,\xi,\eta) - Ay_\infty\left(\frac{p}{2},\xi,\eta\right) = A(\xi - \eta) + c \tag{80}$$

and is the unique solution of the initial value problem (28), (29) *with* H *given by* (30).

We see from Proposition 11 that properties of sequences $x_m(\cdot,\xi,\eta)$, $y_m(\cdot,\xi,\eta)$, $m \geq 0$, constructed for problem (73), (74) are rather similar to those for the periodic problem (1), (2) (in particular, the definition of functions Ξ and H is the same as in Proposition 1). In both cases, the iteration is carried out according to formulae (19), (20), the only difference being in equalities (75), (76) for $x_0(\cdot,\xi,\eta)$ and $y_0(\cdot,\xi,\eta)$. As a result, the corresponding limit functions satisfy the boundary conditions (79), (80).

Based on Proposition 11, one can develop essentially the same techniques that have been indicated above for the periodic problem (1), (2). The main difference in the proofs is that, in addition to guaranteeing that the appropriate values ξ should belong to Ω, we also have to ensure that $A\xi + c \in \Omega$ as well. The convergence of iterations is then guaranteed for all η from the set Ω and ξ belonging to its subset $S_{A,c}(\Omega)$ defined by the relation

$$S_{A,c}(\Omega) := \{\xi \in \Omega : A\xi + c \in \Omega\}. \tag{81}$$

Clearly, $S_{A,c}(\Omega)$ is the union of all the subsets of Ω invariant with respect to the transformation $x \mapsto Ax + c$. For example, if Ω is a set on the plane ($n = 2$) containing the origin, then $S_{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right),0}(\Omega)$ is the part of Ω that is symmetric with respect to the diagonal passing through the first and the third quadrants (see Figure 7).

Theorem 12 *Let there exist a non-negative vector* ϱ *with property* (34) *such that* $f(t,\cdot) \in \operatorname{Lip}_K(\Omega_\varrho)$ *for a.e.* $t \in [0,p]$ *with a certain matrix* K *satisfying inequality* (35). *Then, for*

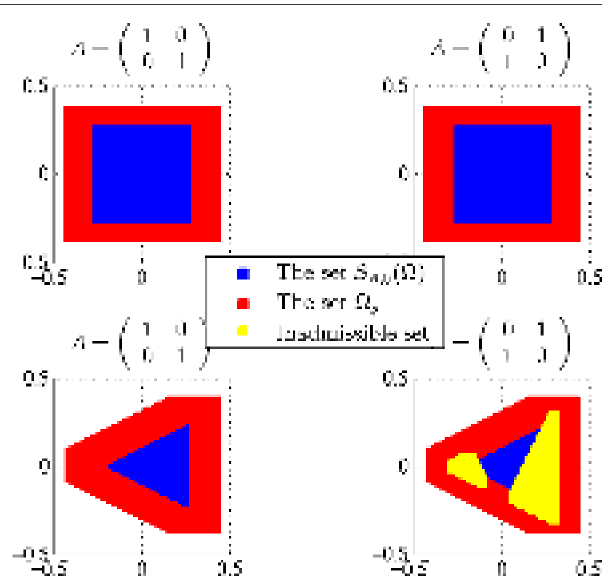

Figure 7 The sets $S_{A,c}(\Omega)$ and Ω_ϱ in examples, $\varrho = \operatorname{col}(1/8, 1/16)$.

all fixed $(\xi, \eta) \in S_{A,c}(\Omega) \times \Omega$, *the sequences* $\{x_m(\cdot, \xi, \eta) : m \geq 0\}$ *and* $\{y_m(\cdot, \xi, \eta) : m \geq 0\}$ *with* $x_0(\cdot, \xi, \eta)$, $y_0(\cdot, \xi, \eta)$ *defined by* (75) *and* (76) *converge uniformly on the corresponding intervals and, moreover, estimates* (36), (37) *hold*.

The same remark as has been made above concerning Theorem 2 applies to Theorem 12: its assertion remains true if (35) is replaced by the inequality

$$r(K) < \frac{2}{\gamma_* p}$$

with γ_* given by (38).

Theorem 12 is easily obtained by analogy to Theorem 2 for the periodic problem. The verification of the conditions of Theorem 12 is also pretty much similar to the latter case. One has to keep in mind that the techniques for the two-point problem (73), (74) are applicable for the values of parameters lying in $S_{A,c}(\Omega) \times \Omega$, and not in the entire Ω^2, which is the case in Theorem 2 (unless A is the unit matrix and $c = 0$). This circumstance has a natural explanation due to (77) and (78), whence one deduces that both ξ and $A\xi + c$ will eventually belong to one and the same set, which fact is then used in Lemma 5. If, for example, c is equal to zero and

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{82}$$

then the assertion of Theorem 12 is true only for the part of Ω that is invariant under the rotation by 45° counter-clockwise. In this way, *e.g.*, Figure 5 is replaced by Figure 8 once the two-point problem (73), (74) with A given by (82) is considered. Note that all the sets on Figures 5 and 8 contain the origin, and the yellow regions on the latter one indicate the points from Ω that cannot be regarded as candidates for initial values of the solution in question.

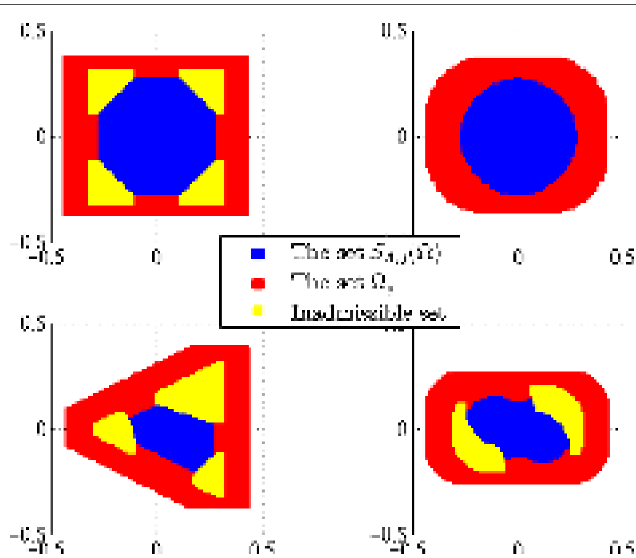

Figure 8 The sets $S_{A,c}(\Omega)$ and Ω_ϱ in examples, $\varrho = \mathrm{col}(1/8, 1/16)$ and A is given by (82).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to the final version of this work and approved its present form.

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A new approach to non-local boundary value problems for ordinary differential systems

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ABSTRACT

We suggest a new constructive approach for the solvability analysis and approximate solution of general non-local boundary value problems for non-linear systems of ordinary differential equations with locally Lipschitzian non-linearities. The practical application of the techniques is explained on a numerical example.

1. Introduction

The purpose of the present note is to provide a scheme for a constructive analysis of a non-local boundary value problem. More precisely, we consider the problem

$$u'(t) = f(t, u(t)), \quad t \in [a, b], \tag{1}$$

$$\phi(u) = d, \tag{2}$$

where $\phi : C([a,b], \mathbb{R}^n) \to \mathbb{R}^n$ is a vector functional (possibly non-linear), $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a function satisfying the Carathéodory conditions in a certain bounded set, and d is a given vector. By a solution of the problem, one means an absolutely continuous function with property (2) satisfying (1) almost everywhere on $[a, b]$.

The analysis is constructive in the sense that, when applicable, it allows one to both study the solvability of the problem and approximately construct its solutions by operating with objects that are determined explicitly in finitely many steps of computation. The topic has been addressed by many authors, see, e.g., [1,2] for related references.

It turns out that, under suitable conditions and with a certain modification, the techniques previously applied in [3,4] for periodic and two-point problems can also be used in the more general cases of problem (1) and (2) where the boundary condition may be non-local. Here, we describe this particular modification, which is based on the introduction of a suitable model problem, and outline the resulting scheme of investigation. Note that the new approach is easier to apply compared with those used earlier, e.g., in [5–7].

2. Notation and symbols

In the sequel, for any $x = \text{col}(x_1, \ldots, x_n) \in \mathbb{R}^n$, the obvious notation $|x| = \text{col}(| x_1 |, \ldots, | x_n |)$ is used and the inequalities between vectors are understood componentwise. A similar convention is adopted implicitly for the operations 'max' and 'min'. The symbol 1_n stands for the unit matrix of dimension n and $r(K)$ denotes the spectral radius of a square matrix K.

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If $z \in \mathbb{R}^n$ and ϱ is a vector with non-negative components, $B(z,\varrho)$ stands for the componentwise ϱ-neighbourhood of z: $B(z,\varrho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leqslant \varrho\}$. Similarly, given a set $\Omega \subset \mathbb{R}^n$, we define its componentwise ϱ-neighbourhood by putting

$$B(\Omega,\varrho) := \bigcup_{z\in\Omega} B(z,\varrho). \tag{3}$$

Given two sets D_0 and D_1 in $\mathbb{R}^n$, we put

$$\mathcal{B}(D_0, D_1) := \{\theta\xi + (1-\theta)\eta : \xi \in D_0, \eta \in D_1, \theta \in [0,1]\}. \tag{4}$$

For a set $\Omega \subseteq \mathbb{R}^n$ and a $n \times n$ matrix K with non-negative entries, we write $f \in \mathrm{Lip}_K(\Omega)$ if the estimate

$$|f(t,u_1) - f(t,u_2)| \leqslant K|u_1 - u_2| \tag{5}$$

holds for all u_1, u_2 from Ω and a.e. $t \in [a,b]$. Finally, we shall frequently use the notation

$$\delta_D(f) := \operatorname*{ess\,sup}_{(t,\xi)\in[a,b]\times D} f(t,\xi) - \operatorname*{ess\,inf}_{(t,\xi)\in[a,b]\times D} f(t,\xi). \tag{6}$$

3. Freezing and parametrization

The idea that we are going to employ is based on the reduction to a family of simpler auxiliary boundary problems obtained by "freezing" certain values of the solution sought for (see, e.g., [8–10]). In our case, the auxiliary problems will have two-point linear separated conditions at a and b:

$$u(a) = \xi, \quad u(b) = \eta, \tag{7}$$

where ξ and η are parameters whose values remain unknown at the moment. As will be seen from the statements below, one can then go back to the original problem by choosing the values of the introduced parameters appropriately.

Let us fix certain bounded sets $D_i \subset \mathbb{R}^n,\ i = 0,1$, and focus on the solutions u of problem (1) and (2) with $u(a) \in D_0$ and $u(b) \in D_1$. Given an arbitrary pair $(\xi,\eta) \in D_0 \times D_1$, we set

$$u_0(t,\xi,\eta) := \left(1 - \frac{t-a}{b-a}\right)\xi + \frac{t-a}{b-a}\eta \tag{8}$$

and

$$u_{m+1}(t,\xi,\eta) = u_0(t,\xi,\eta) + \int_a^t f(s,u_m(s,\xi,\eta))ds - \frac{t-a}{b-a}\int_a^b f(\tau,u_m(\tau,\xi,\eta))d\tau \tag{9}$$

for all $t \in [a,b]$ and $m = 0,1,\ldots$. The vectors ξ and η in (8) and (9) are treated as unknown parameters. Considering formulae (8) and (9), one arrives immediately at the following

Proposition 1. *If, for a fixed pair $(\xi,\eta) \in D_0 \times D_1$, the sequence $\{u_m(\cdot,\xi,\eta) : m \geqslant 0\}$ converges to a function $u_\infty(\cdot,\xi,\eta)$ uniformly on $[a,b]$, then:*

1. $u_\infty(b,\xi,\eta) = \eta$.
2. $u_\infty(\cdot,\xi,\eta)$ *satisfies the Cauchy problem*

$$u'(t) = f(t,u(t)) + \frac{1}{b-a}\Delta(\xi,\eta), \quad t \in [a,b], \tag{10}$$

$$u(a) = \xi, \tag{11}$$

where $\Delta : D_0 \times D_1 \to \mathbb{R}^n$ is given by formula

$$\Delta(\xi,\eta) := \eta - \xi - \int_a^b f(s,u_\infty(s,\xi,\eta))ds. \tag{12}$$

In other words, the function $u_\infty(\cdot,\xi,\eta)$, provided that it is well-defined, satisfies the equation

$$u(t) = u_0(t,\xi,\eta) + \int_a^t f(s,u(s))ds - \frac{t-a}{b-a}\int_a^b f(s,u(s))ds, \quad t \in [a,b]. \tag{13}$$

Since, clearly, the values of $u_0(\cdot,\xi,\eta)$ are convex combinations of ξ and η, we see from (13) that $u_\infty(\cdot,\xi,\eta)$ is also a solution of the two-point boundary problem (10) and (7). It turns out that this simple fact can be used to analyse the solutions of the original problem (1) and (2). In order to continue, it is however necessary to establish conditions ensuring the convergence of sequence (9) and, therefore, the fact that $u_\infty(\cdot,\xi,\eta)$ is well defined for the corresponding values of ξ and η.

4. Convergence of successive approximations

Let us put

$$\Omega := \mathcal{B}(D_0, D_1) \tag{14}$$

and $\Omega_\varrho := B(\Omega, \varrho)$ for any non-negative vector ϱ. Recall that the set $\mathcal{B}(D_0, D_1)$ is defined according to (4).

Remark 2. It is clear from (4) that $\mathcal{B}(D_0, D_1) \subset \operatorname{conv}(D_0 \cup D_1)$ but the equality is, generally speaking, not true.

Theorem 3. *Let there exist a non-negative vector ϱ satisfying the inequality*

$$\varrho \geqslant \frac{b-a}{4}\delta_{\Omega_\varrho}(f), \tag{15}$$

such that $f \in \operatorname{Lip}_K(\Omega_\varrho)$ with a matrix K for which

$$(b-a)r(K) < \frac{1}{\gamma_0}, \tag{16}$$

where

$$\gamma_0 := 3/10. \tag{17}$$

Then, for all fixed $(\xi, \eta) \in D_0 \times D_1$:

1. *The limit $\lim_{m\to\infty} u_m(t,\xi,\eta) =: u_\infty(t,\xi,\eta)$ exists uniformly in $t \in [a,b]$.*
2. *$u_\infty(\cdot,\xi,\eta)$ is the unique solution of the Cauchy problem (10) and (11).*
3. *$u_\infty(t,\xi,\eta) \in \Omega_\varrho$ for any $t \in [a,b]$.*

4. *The estimate*

$$|u_\infty(t,\xi,\eta) - u_m(t,\xi,\eta)| \leqslant \frac{5}{9}\alpha_1(t)(\gamma_0(b-a)K)^m(1_n - \gamma_0(b-a)K)^{-1}\delta_{\Omega_\varrho}(f) \tag{18}$$

holds for any $t \in [a,b]$ and $m \geqslant 0$, where

$$\alpha_1(t) = 2(t-a)\left(1 - \frac{t-a}{b-a}\right), \quad t \in [a,b]. \tag{19}$$

The proof of Theorem 3 is carried out by combining several auxiliary statements given below (see [1,11]).

Lemma 4 [1, Lemma 3.13]. *For any continuous function $u : [a,b] \to \mathbb{R}^n$, the estimate*

$$\left| \int_a^t \left(u(\tau) - \frac{1}{b-a}\int_a^b u(s)ds \right) d\tau \right| \leqslant \frac{1}{2}\alpha_1(t)\omega_{[a,b]}(u), \quad t \in [a,b], \tag{20}$$

holds, where α_1 is given by (19) and $\omega_{[a,b]}(u) := \max_{s\in[a,b]} u(s) - \min_{s\in[a,b]} u(s)$.

Let

$$\alpha_{m+1}(t) := \left(1 - \frac{t-a}{b-a}\right)\int_a^t \alpha_m(s)ds + \frac{t-a}{b-a}\int_t^b \alpha_m(s)ds, \quad t \in [a,b], \tag{21}$$

for any $m \geqslant 0$, where $\alpha_0(t) := 1, t \in [a,b]$. Clearly, formula (19) defining α_1 is obtained from (21) for $m = 0$.

Lemma 5 [1, Lemma 3.16]. *The following estimates hold:*

$$\alpha_{m+1}(t) \leqslant \gamma_0(b-a)\alpha_m(t), \quad t \in [a,b], \tag{22}$$

for $m \geqslant 2$ and

$$\alpha_{m+1}(t) \leqslant \frac{10}{9}(\gamma_0(b-a))^m \alpha_1(t), \quad t \in [a,b], \tag{23}$$

for $m \geqslant 0$, where γ_0 is given by (17).

Lemma 6. *If ϱ is a vector satisfying relation (15), then*

$$\{u_m(t,\xi,\eta) : t \in [a,b]\} \subset \Omega_\varrho \tag{24}$$

for any $m \geqslant 0$ *and* $(\xi, \eta) \in D_0 \times D_1$,

Proof. The proof is analogous to that of [4, Lemma 4] and is based on Lemma 4. Let $(\xi, \eta) \in D_0 \times D_1$ be arbitrary. In view of (14), it follows immediately from (8) that $u_0(t, \xi, \eta) \in \Omega$ for any $t \in [a, b]$, i.e., (24) holds for $m = 0$.

Let us assume that (24) holds for a certain $m = m_0$. Then, by virtue of (9), (15), and Lemma 4, we obtain

$$|u_{m_0+1}(t,\xi,\eta) - u_0(t,\xi,\eta)| \leqslant \frac{b-a}{4}\delta_\Omega(f) \leqslant \varrho \tag{25}$$

for $t \in [a, b]$. Since (24) is known to be true for $m = 0$, we see from (25) that all the values of $u_{m_0+1}(\cdot, \xi, \eta)$ are contained in $B(\Omega, \varrho)$, i.e., (24) holds with $m = m_0 + 1$. The arbitrariness of m_0 then leads us to (24) for any m. □

Proof of Theorem 3. Let $\xi \in D_0$ and $\eta \in D_1$. By Lemma 6, we have $u_m(t, \xi, \eta) \in \Omega_\varrho$ for all $t \in [a, b]$ and $m \geqslant 0$. Since, by assumption, the function f belongs to $\mathrm{Lip}_K(\Omega_\varrho)$, relation (9) yields

$$r_{m+1}(t,z,\eta) \leqslant K\left(\left(1-\frac{t-a}{b-a}\right)\int_a^t r_m(s,z,\eta)ds + \frac{t-a}{b-a}\int_t^b r_m(s,z,\eta)ds\right), \quad t \in [a,b], \tag{26}$$

for all $m \geqslant 1$, where

$$r_m(t,\xi,\eta) := |u_m(t,\xi,\eta) - u_{m-1}(t,\xi,\eta)|, \quad t \in [a,b], \ m \geqslant 1. \tag{27}$$

On the other hand, using (9) and Lemma 4, we obtain

$$\begin{aligned} r_1(t,\xi,\eta) &= \left|\int_a^t \left(f(s,u_0(s,\xi,\eta)) - \frac{1}{b-a}\int_a^b f(s,u_0(\tau,\xi,\eta))d\tau\right)ds\right| \\ &\leqslant \frac{1}{2}\alpha_1(t)\omega_{[a,b]}(f(\cdot,u_0(\xi,\eta))) \\ &\leqslant \frac{1}{2}\alpha_1(t)\delta_{\Omega_\varrho}(f) \end{aligned} \tag{28}$$

for any $t \in [a, b]$. Putting in (26) $m = 1$ and using (21) and estimate (23) of Lemma 5, we obtain

$$\begin{aligned} r_2(t,\xi,\eta) &\leqslant \frac{1}{2}K\left(\left(1-\frac{t-a}{b-a}\right)\int_a^t \alpha_1(s)ds + \frac{t-a}{b-a}\int_t^b \alpha_1(s)ds\right)\delta_{\Omega_\varrho}(f) \\ &\leqslant \frac{1}{2}K\alpha_2(t)\delta_{\Omega_\varrho}(f) \\ &\leqslant \frac{5\gamma_0}{9}K\alpha_1(t)\delta_{\Omega_\varrho}(f), \end{aligned} \tag{29}$$

where γ_0 is given by (17). Considering (26) and (29) and arguing by induction, we conclude that

$$r_{m+1}(t,\xi,\eta) \leqslant \frac{1}{2}K^m\alpha_{m+1}(t)\delta_{\Omega_\varrho}(f) \leqslant \frac{5}{9}(\gamma_0(b-a)K)^m\alpha_1(t)\delta_{\Omega_\varrho}(f), \quad t \in [a,b], \tag{30}$$

for any $m \geqslant 0$. Therefore, using (19) and the equality $\max_{s\in[a,b]}\alpha_1(s) = \frac{1}{2}(b-a)$, we get

$$\begin{aligned} |\ u_{m+j}(t,\xi,\eta) - u_m(t,\xi,\eta)\ | &\leqslant \sum_{i=1}^{j} r_{m+i}(t,\xi,\eta) \\ &\leqslant \frac{5}{9}\alpha_1(t)\sum_{i=1}^{j}(\gamma_0(b-a)K)^{m+i-1}\delta_{\Omega_\varrho}(f) \\ &\leqslant \frac{5(b-a)}{18}(\gamma_0(b-a)K)^m\sum_{i=0}^{j-1}(\gamma_0(b-a)K)^i\delta_{\Omega_\varrho}(f) \end{aligned} \tag{31}$$

for any $t \in [a, b], m \geqslant 0$, and $j \geqslant 1$. In view of assumption (16), the sums involved in (31) are bounded and $\lim_{m\to\infty}(\gamma_0(b-a)K)^m = 0$. Therefore, (31) implies that $\{u_m(\cdot, \xi, \eta) : m \geqslant 0\}$ is a Cauchy sequence in $C([a, b], \mathbb{R}^n)$. Passing to the limit as $j \to \infty$ in (31), one arrives at (18).

5. Properties of the function $u_\infty(\cdot, \xi, \eta)$

In terms of function $u_\infty(\cdot, \xi, \eta)$, one can characterise the solvability of the two-point problem with separated conditions (7). More precisely, apart of system (1), consider the forced system

$$u'(t) = f(t, u(t)) + \mu(b-a)^{-1}, \quad t \in [a,b], \tag{32}$$

where $\mu = \mathrm{col}(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ is a control parameter.

Theorem 7. *Let $\xi \in D_0$ and $\eta \in D_1$ be fixed. Let there exist a non-negative vector ϱ wit property (15) such that $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with a matrix K for which (16) holds. Then, for the solution of system (32) with*

$$u(a) = \xi \tag{33}$$

to have the property

$$u(b) = \eta, \tag{34}$$

it is necessary and sufficient that

$$\mu = \Delta(\xi, \eta), \tag{35}$$

where $\Delta(\xi,\eta)$ is given by (12). Moreover, in the case where (35) holds, the solution of the initial value problem (32) and (33) coincides with $u_\infty(\cdot, \xi, \eta)$.

In other words, for any given pair (ξ, η), the vector $\Delta(\xi,\eta)$ is the only value of μ in (32) for which the solution of (32) and (33) satisfies the two-point boundary conditions (7).

Proof of Theorem 7. *Sufficiency*. Assume that (35) holds. In that case, (10) coincides with (32). By virtue of Proposition 1, the function $u_\infty(\cdot,\xi,\eta)$ is the unique solution of the initial value problem (10), (11) and, moreover, $u_\infty(b,\xi,\eta) = \eta$. Thus, $u_\infty(\cdot,\xi,\eta)$ is a solution of (32) and (34).

Necessity. Let $u_\mu(\cdot,\xi)$ denote the solution of the initial value problem (32) and (33). It is obvious from (32) and (33) that

$$u_\mu(t,\xi) = \xi + \int_a^t f(s, u_\mu(s,\xi))ds + \mu\frac{t-a}{b-a}, \quad t \in [a,b]. \tag{36}$$

It follows immediately from (36) that the value of μ can be represented as

$$\mu = u_\mu(b,\xi) - \xi - \int_a^b f(s, u_\mu(s,\xi))ds \tag{37}$$

and, therefore,

$$u_\mu(t,\xi) = \xi + \int_a^t f(s,u_\mu(s,\xi))ds + \frac{t-a}{b-a}\left(u_\mu(b,\xi) - \xi - \int_a^b f(s,u_\mu(s,\xi))ds\right), \quad t \in [a,b], \tag{38}$$

for any μ. In particular, $u_{\Delta(\xi,\eta)}(\cdot,\xi)$ satisfies the equation

$$u_{\Delta(\xi,\eta)}(t,\xi) = \xi + \int_a^t f(s,u_{\Delta(\xi,\eta)}(s,\xi))ds + \frac{t-a}{b-a}\left(\eta - \xi - \int_a^b f(s,u_{\Delta(\xi,\eta)}(s,\xi))ds\right), \quad t \in [a,b], \tag{39}$$

since, in view of Proposition 1, $u_{\Delta(\xi,\eta)}(\cdot,\xi)$ coincides with $u_\infty(\cdot,\xi,\eta)$ and the latter function has the property $u_\infty(b,\xi,\eta) = \eta$. Assuming now that

$$u_\mu(b,\xi) = \eta, \tag{40}$$

we immediately find from (38) and (39) that each of the functions $u_\mu(\cdot,\xi)$ and $u_{\Delta(\xi,\eta)}(\cdot,\xi)$ satisfies Eq. (13), where $u_0(\cdot,\xi,\eta)$ is given by (8). By Theorem 3, the function $u_\infty(\cdot,\xi,\eta)$, which is the uniform limit of the successive approximations (9), is the only solution of (13). Therefore, under assumption (40), $u_\mu(\cdot,\xi)$ coincides with $u_\infty(\cdot,\xi,\eta)$. Recalling (37), we conclude that μ necessarily has form (35) in that case.

Theorem 7 leads one immediately to the following

Proposition 8. *Under the assumptions of Theorem 3, the function $u_\infty(\cdot,\xi,\eta)$ is a solution of the boundary value problem (1) and (2) if and only if the pair (ξ,η) satisfies the system of $2n$ equations*

$$\Delta(\xi,\eta) = 0, \tag{41}$$

$$\phi(u_\infty(\cdot,\xi,\eta)) = d, \tag{42}$$

where $\Delta : D_0 \times D_1 \to \mathbb{R}^n$ is given by (12).

Proof. It suffices to apply Theorem 7 and notice that the differential Eq. (10) coincides with (1) if and only if (ξ,η) satisfies (41). □

Equations of the type appearing in the last proposition are usually referred to as a *determining equations* and, indeed, as the following statement shows, the system of Eqs. (41) and (42) determines all possible solutions of the original boundary value problem (1), (2) with graphs lying in Ω_ϱ.

Theorem 9. *Let there exist a non-negative vector ϱ with property (15) such that $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with a matrix K for which (16) holds.*

1. *If there exists a pair $(\xi, \eta) \in D_0 \times D_1$ satisfying (41) and (42), then the non-local boundary value problem (1) and (2) has a solution $u(\cdot)$ such that*
$$\{u(t) : t \in [a, b]\} \subset \Omega_\varrho \tag{43}$$
and $u(a) = \xi, u(b) = \eta$.
2. *If the boundary value problem (1) and (2) has a solution $u(\cdot)$ such that (43) holds, then the pair $(u(a), u(b))$ is a solution of system (41) and (42).*

Proof. The first assertion is an immediate consequence of Theorem 3 and Propositions 1 and 8 since $u_\infty(\cdot, \xi, \eta)$ is the required solution in that case. To prove the second one, assume that problem (1) and (2) has a solution u with property (43). Then u is a solution of the Cauchy problem (32) and (33) with $\mu = 0$ and $\xi = u(a)$ and, therefore, by Theorem 3,
$$u = u_\infty(\cdot, u(a), u(b)). \tag{44}$$
In view of Theorem 7, we obtain
$$\Delta(\xi, u(b)) = 0, \tag{45}$$
which means that (41) holds with $\eta = u(b)$. Finally, equality (42) is an immediate consequence of (44) and the assumption that $\phi(u) = d$. □

6. Approximation of a solution

The last theorem suggests an approach to the study of the non-local problem (1) and (2) by looking for its solution among those of the family of equations (13), which are, as Theorem 7 shows, motivated by auxiliary problems with separated two-point conditions (7). The study of the problem then consists of two parts, namely, the analytic part, when the integral Eq. (13) is dealt with by using the method of successive approximations (9), and the numerical one, which consists in finding a values of the $2n$ unknown parameters from the system of Eqs. (41) and (42). This closely correlates with the idea of the Lyapunov–Schmidt reduction (see, e.g., [12]). The solvability of the determining system (41) and (42), in turn, can be established in a rigorous manner by studying some its approximate versions
$$\Delta_m(\xi, \eta) = 0, \tag{46}$$
$$\phi(u_m(\cdot, \xi, \eta)) = d, \tag{47}$$
where m is fixed and $\Delta_m : D_0 \times D_1 \to \mathbb{R}^n$ is given by the relation
$$\Delta_m(\xi, \eta) := \eta - \xi - \int_a^b f(s, u_m(s, \xi, \eta))ds \tag{48}$$
for all $(\xi, \eta) \in D_0 \times D_1$. The solvability analysis based on properties of equations (46) and (47), which can be carried out by analogy to [3,4,13], is not treated here.

In practice, one constructs analytically the function $u_{m_0}(\cdot, \xi, \eta)$ for a certain m_0 keeping ξ and η as parameters, then finds numerically a root $(\tilde{\xi}, \tilde{\eta})$ of the approximate determining system (46) and (47) with $m = m_0$, and forms the function
$$U_{m_0}(t) := u_{m_0}(t, \tilde{\xi}, \tilde{\eta}), \quad t \in [a, b], \tag{49}$$
which is natural to be interpreted as the m_0th approximation of a solution of the original problem (1) and (2) the values of which at a and b lie in a neighbourhood of $\tilde{\xi}$ and $\tilde{\eta}$ respectively. Possible multiple roots of system (46) and (47), under appropriate assumptions, correspond to multiple solutions of the exact determining system (41), (42) and, thus, determine distinct solutions of the given problem.

The above-mentioned property of U_{m_0} is justified by the estimate
$$|u_\infty(t, \tilde{\xi}, \tilde{\eta}) - U_{m_0}(t)| \leqslant \frac{5}{9}\alpha_1(t)(\gamma_0(b-a)K)^{m_0}(1_n - (\gamma_0(b-a)K))^{-1}\delta_{\Omega_\varrho}(f), \quad t \in [a, b], \tag{50}$$
which is a direct consequence of inequality (18) of Theorem 3. In (50), ϱ is the vector appearing in Theorem 3, whereas $\delta_{\Omega_\varrho}(f)$ and γ_0 are given by (6) and (17) respectively. Note that, by Theorem 9, a solution of problem (1) and (2), when it exists,

necessarily has the form $u_\infty(\cdot,\xi_*,\eta_*)$, where (ξ_*,η_*) satisfies (41) and (42). The pair $(\tilde{\xi},\tilde{\eta})$ involved in (50) is, in a sense, an approximation of the explicitly unknown (ξ_*,η_*). A rigorous proof of the existence of the solution in question would involve an analysis of the approximate determining Eqs. (46) and (47) in the spirit of [4,13].

The most difficult part of the scheme is, of course, the construction of the function $u_{m_0}(\cdot,\xi,\eta)$. Quite often systems of symbolic computation can be used for this purpose, which facilitates greatly the operations with functions depending on multiple parameters. Otherwise, if the explicit integration in the (9) if impossible or difficult, one employs suitable modifications of the formulae which, at the expense of a certain loss in accuracy, lead one to schemes better suited for practical realisation. We mention two natural modifications of this kind which make the scheme more constructive.

Version 1 ("Frozen" parameters) Instead of $\{u_m : m \geqslant 0\}$ defined by (9), one uses the sequence $\{v_m : m \geqslant 0\}$ defined by the equalities

$$v_0(t,\xi,\eta) := u_0(t,\xi,\eta), \quad t \in [a,b], \tag{51}$$

and

$$v_{m+1}(t,\xi,\eta) := u_0(t,\xi,\eta) + \int_a^t f(s, v_m(s,\xi_m,\eta_m))ds - \frac{t-a}{b-a}\int_a^b f(\tau, v_m(\tau,\xi_m,\eta_m))d\tau, \quad t \in [a,b], \tag{52}$$

for any $m = 0,1,\ldots$, where $u_0(\cdot,\xi,\eta)$ is given by (8) and (ξ_m,η_m) is a root of the system

$$\begin{aligned} \eta - \xi &= \int_a^b f(s, v_m(s,\xi,\eta))ds, \\ \phi(v_m(\cdot,\xi,\eta)) &= d. \end{aligned} \tag{53}$$

Then one defines the function U_{m_0}, which is to be treated as the m_0th approximation of a solution u with $(u(a),u(b))$ lying in a neighbourhood of (ξ_{m_0},η_{m_0}), as

$$U_{m_0}(t) := v_{m_0}(t,\xi_{m_0},\eta_{m_0}), \quad t \in [a,b]. \tag{54}$$

Note that, as follows from (51) and (52), the mapping $(\xi,\eta) \mapsto v_m(t,\xi,\eta)$ is linear for any $t \in [a,b]$ and, moreover, the dependence on the parameters in (52) is localised to the first summand outside the integration sign. This facilitates greatly the construction of iterations compared to formula (9). For the same reason, system (53), which has to be solved numerically, is considerably simpler than (46) and (47).

System (53) should be solved in a domain where the values $(u(a),u(b))$ of a solution are expected to lie. A natural starting point for that is a root (ξ_0,η_0) of the zeroth approximate determining system ((46) and (47) with $m = 0$):

$$\begin{aligned} \eta - \xi &= \int_a^b f(s, u_0(s,\xi,\eta))ds, \\ \phi(u_0(\cdot,\xi,\eta)) &= d, \end{aligned} \tag{55}$$

where u_0 is given by (8).

Version 2 (Polynomial interpolation) Formula (52) is modified so that the polynomial approximations of the integrands are used, i.e., instead of (9), one uses the formula

$$v_{m+1}(t,\xi,\eta) := u_0(t,\xi,\eta) + \int_a^t p_l f(\cdot, v_m(\cdot,\xi_m,\eta_m))(s)ds - \frac{t-a}{b-a}\int_a^b p_l f(\cdot, v_m(\cdot,\xi_m,\eta_m))(\tau)d\tau, \quad t \in [a,b],$$

where l is fixed and $p_l y$ stands for the polynomial of degree l interpolating the function y at l suitably chosen nodes. The substantiation is similar to other similar cases (see, e.g., [14] where Dirichlet problems for systems of two equations are considered). In this case, one assumes that f satisfies the Dini condition in the time variable [15].

Combining Versions 1 and 2 and using computer algebra systems to facilitate the computation, one arrives at a scheme which is quite efficient and easy to be programmed.

7. A numerical example

Let us apply the numerical-analytic approach described above to the system of differential equations

$$\begin{aligned} u_1'(t) &= u_2^2(t) - \frac{t}{5}u_1(t) + \frac{t^3}{100} - \frac{t^2}{25}, \\ u_2'(t) &= \frac{t^2}{10}u_2(t) + \frac{t}{8}u_1(t) - \frac{21}{800}t^3 + \frac{1}{16}t + \frac{1}{5}, \quad t \in [0,1/2], \end{aligned} \tag{56}$$

considered under the non-linear boundary conditions of integral type

$$\int_0^{\frac{1}{2}} su_1(s)u_2(s)ds = -\frac{197}{48000}, \quad \int_0^{\frac{1}{2}} s^2u_2^2(s)ds = \frac{1}{4000}. \tag{57}$$

Problem (56) and (57) has form (1) and (2) with $a = 0, b = 1/2$,

$$u \mapsto \phi(u) := \begin{pmatrix} \int_0^{\frac{1}{2}} s u_1(s) u_2(s) ds \\ \int_0^{\frac{1}{2}} s^2 u_2^2(s) ds \end{pmatrix},$$

and the obvious definitions of the function $f : [0, 1/2] \times \mathbb{R}^2 \to \mathbb{R}^2$ and vector d.

We need to choose some domains where the values of a solution at 0 and 1/2 should belong. Let us put, e.g.,

$$D_0 := \{(u_1, u_2) : -0.55 \leqslant u_1 \leq 0.45, -0.2 \leqslant u_2 \leq 0.15\}, \quad D_1 := D_0. \tag{58}$$

It is clear from (4) that $\mathcal{B}(D_0, D_0) = D_0$ and, therefore, according to (14), we have $\Omega = D_0$ in this case. Putting

$$\varrho := \text{col}(0.2, 0.2), \tag{59}$$

we find that the componentwise ϱ-neighbourhood of the set Ω has the form

$$\Omega_\varrho = \{(u_1, u_2) : -0.75 \leqslant u_1 \leq 0.65, -0.4 \leqslant u_2 \leq 0.35\} \tag{60}$$

and, according to (6), one gets that $\delta_{\Omega_\varrho}(f) = \text{col}(0.3, 0.10625)$.

Therefore,

$$\frac{b-a}{4}\delta_{\Omega_\varrho}(f) = \begin{pmatrix} 0.0375 \\ 0.01328125 \end{pmatrix} \leqslant \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix} = \varrho, \tag{61}$$

which means that the value of ϱ given by (59) satisfies inequality (15) of Theorem 3. A direct computation also shows that $f \in \text{Lip}_K(\Omega_\varrho)$ with

$$K := \begin{pmatrix} 1/10 & 9/10 \\ 1/16 & 1/40 \end{pmatrix}$$

and, therefore,

$$\frac{3}{20} r(K) = 0.045 < 1, \tag{62}$$

which means that (16) holds. We see that all the conditions of Theorem 3 are satisfied. The sequence of functions (9) is thus convergent and one can continue to the construction of approximations.

According to Theorem 9, the number of roots of the determining system (41) and (42) in $D_0 \times D_1$ coincides with the number of solutions u of problem (56) and (57) with $\{u(0), u(1/2)\}$ lying in the set (60). The approximate determining systems (46) and (47) are regarded as approximations to (41), (42) and, thus, their roots may serve as approximations to those of (41) and (42). Let us consider several approximations of a concrete solution.

We start from the zeroth approximation, in which case no iteration is carried out at all. Formula (8) in this example gives

$$u_{0i}(t, \xi, \eta) = (1 - 2t)\xi_i + 2t\eta_i \tag{63}$$

for $i = 1, 2$, where $u_m = \text{col}(u_{m1}, u_{m2}), m \geqslant 0$. Substituting (63) into (48), we find that the zeroth approximate determining system $\Delta_0(\xi, \eta) = 0$ in this case has the form

$$\begin{aligned} &-\frac{119}{60}\xi_1 + \frac{61}{30}\eta_1 - \frac{1}{6}(\eta_2 - \xi_2)^2 - \xi_2(\eta_2 - \xi_2) - \xi_2^2 + \frac{29}{9600} = 0, \\ &-\frac{961}{480}\xi_2 + \frac{319}{160}\eta_2 - \frac{1}{96}\xi_1 - \frac{1}{48}\eta_1 - \frac{5499}{25600} = 0. \end{aligned} \tag{64}$$

It is easy to verify that the pair of functions

$$u_1^*(t) = \frac{t^2}{20} - \frac{1}{2}, \quad u_2^*(t) = \frac{t}{5}, \quad t \in [0, 1/2], \tag{65}$$

is a solution of problem (56) and (57). Obviously, $(u_1^*(0), u_2^*(0)) = (\xi_1^*, \xi_2^*)$ and $(u_1^*(1/2), u_2^*(1/2)) = (\eta_1^*, \eta_2^*)$ with

$$\begin{aligned} &\xi_1^* = -0.5, \quad \xi_2^* = 0, \\ &\eta_1^* = -0.4875, \quad \eta_2^* = 0.1. \end{aligned} \tag{66}$$

Solving the system of Eqs. (64) in a neighbourhood of the point $(-0.5, 0, 0.4875, 0.1)$, we find its root $(\xi_{01}, \xi_{02}, \eta_{01}, \eta_{02})$:

$$\begin{aligned} &\xi_{01} \approx -0.5018743329, \quad \xi_{02} \approx -0.2568969557 \cdot 10^{-5} \\ &\eta_{01} \approx -0.4893794933, \quad \eta_{02} \approx 0.1000006422 \end{aligned} \tag{67}$$

and, after the substitution of (67) into (63), obtain the corresponding zeroth approximation $U_0 = u_0(\cdot, \xi_0, \eta_0)$ of solution (65):

$$\begin{aligned} U_{01}(t) &\approx -0.5018743329 + 0.0249896794\,t, \\ U_{02}(t) &\approx -0.000002568969557 + 0.2000064223\,t, \quad t \in [0, 1/2], \end{aligned} \tag{68}$$

shown on Fig. 1. Here and below, we use the notation $U_m = \mathrm{col}(U_{m1}, U_{m2})$, $\xi_m = \mathrm{col}(\xi_{m1}, \xi_{m2})$, $\eta_m = \mathrm{col}(\eta_{m1}, \eta_{m2})$ for any m.

According to (8) and (49), the zeroth approximation is always a linear function and, therefore, one cannot expect a satisfactory degree of accuracy at the very beginning of computation (see the graphs of u_1^* and U_{01} at Fig. 1(a)). However, values (67) can already serve as approximations of (66) and, thus, even the zeroth approximate determining system (64) helps us to obtain a certain space localisation of the corresponding roots of the approximate determining systems at further steps. Indeed, let us construct the first approximation. Using (9) and carrying out computations in *Maple*, at the first iteration ($m = 1$), we obtain

$$\begin{aligned} u_{11}(t,\xi,\eta) &= \xi_1 + \frac{t^4}{400} + \frac{t^3}{3}\left(4(-\xi_2+\eta_2)^2 + \frac{2}{5}(\xi_1 - \eta_1) - \frac{1}{25}\right) \\ &\quad + \frac{t^2}{2}\left(4\xi_2(-\xi_2+\eta_2) - \frac{1}{5}\xi_1\right) + \xi_2^2 t \\ &\quad - 2t\left(-\frac{29}{19200} + \frac{1}{6}(-\xi_2+\eta_2)^2 - \frac{1}{120}\xi_1 - \frac{1}{60}\eta_1 + \frac{1}{2}\xi_2(-\xi_2+\eta_2) + \frac{1}{2}\xi_2^2\right) \\ &\quad + 2t(\eta_1 - \xi_1), \\ u_{12}(t,\xi,\eta) &= \xi_2 + \frac{t}{5} + \frac{t^4}{20}\left(-\xi_2 + \eta_2 - \frac{21}{160}\right) + \frac{t^3}{6}\left(-\frac{1}{2}\xi_1 + \frac{1}{2}\eta_1 + \frac{1}{5}\xi_2\right) + \frac{t^2}{16}\left(\xi_1 + \frac{1}{2}\right) \\ &\quad - \frac{t}{16}\left(\frac{5499}{1600} + \frac{1}{30}\xi_2 + \frac{1}{10}\eta_2 + \frac{1}{6}\xi_1 + \frac{1}{3}\eta_1\right) + 2t(\eta_2 - \xi_2) \end{aligned} \tag{69}$$

for any $t \in [0, 1/2]$ and $\{\xi, \eta\} \subset D_0$. Solving numerically the approximate determining system (46) and (47) for $m = 1$ in a neighbourhood of $(\xi_{01}, \xi_{02}, \eta_{01}, \eta_{02})$, we find its root $(\xi_{11}, \xi_{12}, \eta_{11}, \eta_{12})$:

$$\begin{aligned} \xi_{11} &\approx -0.5000145056, \quad \xi_{12} \approx 5.750026703 \cdot 10^{-7}, \\ \eta_{11} &\approx -0.4875143149, \quad \eta_{12} \approx 0.1000004007. \end{aligned} \tag{70}$$

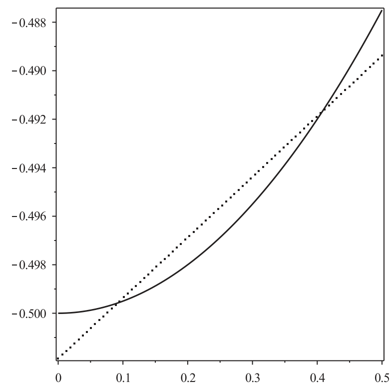
Recall that $(\xi_{01}, \xi_{02}, \eta_{01}, \eta_{02})$ is the root (67) of system (64). Using (49) and substituting the values (70) into (69), we obtain the first and second components of the first approximation $U_1 = \mathrm{col}(U_{11}, U_{12})$ of the solution of problem (56) and (57):

$$\begin{aligned} U_{11}(t) &= -0.5000145056 + \frac{t^4}{400} - 0.001666738533\,t^3 + 0.05000156555\,t^2 + 0.00010378326\,t, \\ U_{12}(t) &= 5.750026703 \cdot 10^{-7} + 0.1999349926\,t - 0.001562508715\,t^4 + 0.001041701733\,t^3 - 9.066 \cdot 10^{-7} t^2 \end{aligned} \tag{71}$$

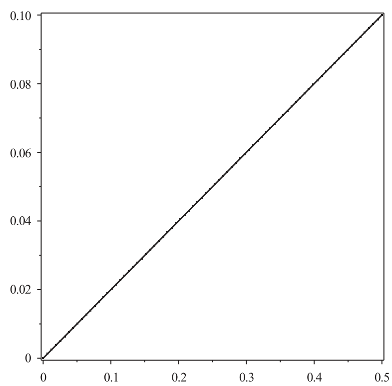
for $t \in [0, 1/2]$. Comparing (71) with (65), we find that the error of the first approximation is estimated as

$$\max_{t\in[0,1/2]} |u_1^*(t) - U_{11}(t)| \leqslant 2 \cdot 10^{-5}, \quad \max_{t\in[0,1/2]} |u_2^*(t) - U_{12}(t)| \leqslant 6 \cdot 10^{-6}. \tag{72}$$

The graphs of the solution (65) and its first approximation are shown on Fig. 2. Considering estimates (72), we see that, in fact, there is no need to draw the graphs of any higher approximations.

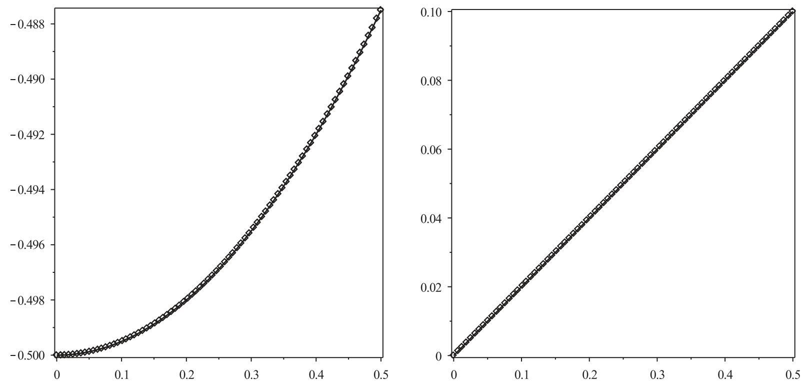


(a) First component



(b) Second component

Fig. 1. The exact solution u^* (solid line) and its zeroth approximation U_0 (dots).



(a) First component (b) Second component

Fig. 2. Solution (65) and its first approximation (71). The graphs of the components of U_1 (the symbol '◇') visually coincide with those of u^* drawn with the solid line.

In case a better accuracy is needed, higher approximations can be constructed in a similar manner. For example, solving the second approximate determining system (46) and (47) ($m = 2$), we obtain the roots

$$\begin{aligned} \xi_{21} &\approx -0.4999999582, \quad \xi_{22} \approx -1.145685349 \cdot 10^{-8}, \\ \eta_{21} &\approx -0.4874999580, \quad \eta_{22} \approx 0.09999998851 \end{aligned} \tag{73}$$

and the corresponding second approximation $U_2 = \mathrm{col}(U_{21}, U_{22})$ of the form

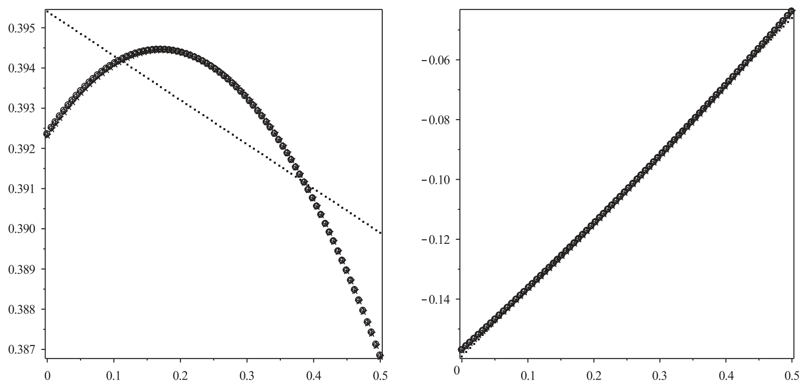
$$\begin{aligned} U_{21}(t) = {} & -0.4999999582 + 2.712673614 \cdot 10^{-7} t^9 - 4.06900898 \cdot 10^{-7} t^8 \\ & + 1.550086457 \cdot 10^{-7} t^7 - 0.00018746609 t^6 + 0.0001499728534 t^5 + 5.7900 \cdot 10^{-10} t^4 \\ & - 0.00001562404305 t^3 + 0.04999999353 t^2 + 3.9424 \cdot 10^{-7} t, \\ U_{22}(t) = {} & -1.145685349 \cdot 10^{-8} + 0.1999998999 t - 0.00002232142859 t^7 \\ & + 0.0000694444383 t^6 - 0.00004166661640 t^5 - 0.000001627838355 t^4 \\ & + 0.00000434008107 t^3 + 2.61 \cdot 10^{-9} t^2 \end{aligned} \tag{74}$$

for all $t \in [0, 1/2]$. We see that (73) as an approximation of (66) is more accurate than (70). A further computation leads to the uniform estimates

$$\max_{t \in [0,1/2]} | u_1^*(t) - U_{21}(t) | \leqslant 6 \cdot 10^{-8}, \quad \max_{t \in [0,1/2]} | u_2^*(t) - U_{22}(t) | \leq 1.5 \cdot 10^{-8},$$

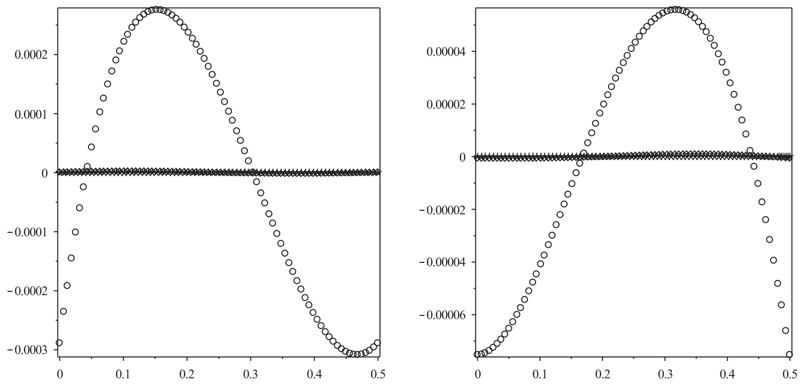
which are significantly better than (72) for U_1. The third approximation U_3, not given here explicitly, provides still better accuracy:

$$\max_{t \in [0,1/2]} | u_1^*(t) - U_{31}(t) | \leq 8 \cdot 10^{-10}, \quad \max_{t \in [0,1/2]} | u_2^*(t) - U_{32}(t) | \leq 1.5 \cdot 10^{-10}.$$



(a) First component (b) Second component

Fig. 3. The zeroth ('·'), first ('∘'), second ('◇') and third ('∗') approximations to $\tilde{u}$.



(a) First component (b) Second component

Fig. 4. The residual functions of the first ('$\circ$'), second ('$\diamond$') and third ('$*$') approximations to $\tilde{u}$.

Let us now note that the considerations shown above have been related to the approximation of solution (65), which is known explicitly in this particular example. A computation shows however that, along with (67), the zeroth approximate determining system (64) has another root

$$\tilde{\xi}_{01} \approx 0.3954059502, \quad \tilde{\xi}_{02} \approx -0.1592025648, \\ \tilde{\eta}_{01} \approx 0.389899163, \quad \tilde{\eta}_{02} \approx -0.0459889168 \tag{75}$$

and, likewise, the first approximate determining system, along with its root (70), has the root

$$\tilde{\xi}_{11} \approx 0.3923536713, \quad \tilde{\xi}_{12} \approx -0.1570525052, \\ \tilde{\eta}_{11} \approx 0.386849396, \quad \tilde{\eta}_{12} \approx -0.04383992217. \tag{76}$$

This indicates a possible existence of a solution $\tilde{u} = \mathrm{col}(\tilde{u}_1, \tilde{u}_2)$ of the boundary value problem (56) and (57) which is different from (65) and has the initial data $(\tilde{u}_1(0), \tilde{u}_2(0), \tilde{u}_1(1/2), \tilde{u}_2(1/2))$ in a neighbourhood of the corresponding values (76). The rigorous analysis confirming the existence of $\tilde{u}$, which consists in the verification of suitable sufficient conditions similar to [3], is omitted here, and we focus on the construction of approximations only. In this case, arguing as shown above and substituting the values from (76) into (69), we obtain the following expression for the first approximation to $\tilde{u}$:

$$\widetilde{U}_{11}(t) = 0.3923536713 + 0.0025\,t^4 + 0.004490021967\,t^3 - 0.07479600670\,t^2 + 0.02495444725\,t,$$
$$\widetilde{U}_{12}(t) = -0.1570525052 + 0.200075291\,t - 0.0009018708475\,t^4 - 0.005693773113\,t^3 + 0.05577210445\,t^2$$

for $t \in [0, 1/2]$. Solving the second determining system in a neighbourhood of $(\tilde{\xi}_{11}, \tilde{\xi}_{12}, \tilde{\eta}_{11}, \tilde{\eta}_{12})$, we find

$$\tilde{\xi}_{21} \approx 0.3923271761, \quad \tilde{\xi}_{22} \approx -0.1570509845, \\ \tilde{\eta}_{21} \approx 0.3868231849, \quad \tilde{\eta}_{22} \approx -0.04383842534, \tag{77}$$

and obtain the second approximation $\widetilde{U}_2$ of $\tilde{u}$:

$$\begin{aligned} \widetilde{U}_{21}(t) &= 0.3923271761 + 9.037479779 \cdot 10^{-8} t^9 + 0.000001283746929\,t^8 \\ &\quad - 0.000009739630169\,t^7 - 0.0002493277103\,t^6 + 0.0000434563574\,t^5 + 0.01226591636\,t^4 \\ &\quad - 0.00749262869\,t^3 - 0.07065485865\,t^2 + 0.02466458183\,t, \\ \widetilde{U}_{22}(t) &= -0.1570509845 + 0.2000007654\,t - 0.00001288388631\,t^7 - 0.00004281164578\,t^6 \\ &\quad + 0.001227658392t^5 - 0.0038978801\,t^4 - 0.004195302573\,t^3 + 0.0557704485\,t^2 \end{aligned} \tag{78}$$

for $t \in [0, 1/2]$. Similarly, one finds the root of the third approximate determining system

$$\tilde{\xi}_{31} \approx 0.3923269706, \quad \tilde{\xi}_{32} \approx -0.1570509371, \\ \tilde{\eta}_{31} \approx 0.3868229824, \quad \tilde{\eta}_{32} \approx -0.04383837836 \tag{79}$$

and constructs the third approximation $\widetilde{U}_3$:

$$\begin{aligned}
\widetilde{U}_{31}(t) = {} & 0.3923269706 + 1.106630226 \cdot 10^{-11} t^{15} + 7.879714253 \cdot 10^{-11} t^{14} \\
& - 2.29239859 \cdot 10^{-9} t^{13} - 3.896994250 \cdot 10^{-10} t^{12} + 1.755383703 \cdot 10^{-7} t^{11} \\
& - 0.00000109051396\, t^{10} - 3.431360411 \cdot 10^{-7} t^{9} + 0.00002580341034\, t^{8} \\
& + 0.00001123554886\, t^{7} - 0.0008109829273\, t^{6} + 0.0008310140856\, t^{5} \\
& - 0.0003399644806\, t^{5} + 0.00634405846\, t^{4} - 0.00149463484\, t^{3} - 0.0694077454\, t^{2} \\
& + 0.02241968655\, t + 0.01193924453\, t^{4} - 0.007483401567\, t^{3} - 0.07064300545\, t^{2} \\
& + 0.02466500215\, t, \\
\widetilde{U}_{32}(t) = {} & -0.1570509371 + 0.200000005\, t + 1.026986386 \cdot 10^{-9} t^{11} - 1.12792034 \cdot 10^{-7} t^{10} \\
& - 6.109572563 \cdot 10^{-7} t^{9} + 0.00001144998193\, t^{8} - 0.00005490798177\, t^{7} + 0.0001856181722\, t^{6} \\
& + 0.000928093109\, t^{5} - 0.00377044416\, t^{4} - 0.004207340707\, t^{3} + 0.05577043565\, t^{2}
\end{aligned} \tag{80}$$

for $t \in [0, 1/2]$.

The graphs of the functions $U_{mi}, m = 1,2,3, i = 1,2$, presented on Fig. 3, show a clear tendency of convergence to $\tilde{u}_i, i = 1,2$. Substituting the third approximation (80) into the differential system (56), one obtains a residual such that

$$\max_{t\in[0,1/2]} \left| \widetilde{U}'_{31}(t) - \widetilde{U}^2_{32}(t) + \frac{t}{5}\widetilde{U}_{31}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| \approx 1.530806 \cdot 10^{-9},$$

$$\max_{t\in[0,1/2]} \left| \widetilde{U}'_{32}(t) - \frac{t^2}{10}\widetilde{U}_{32}(t) + \frac{t^3}{50} - \frac{1}{5} \right| \approx 9.9868 \cdot 10^{-11},$$

whereas the residual of the first approximation U_1 does not exceed 0.0004 (see also Fig. 4).

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On non-linear boundary value problems and parametrisation at multiple nodes

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. We show how a suitable interval division and parametrisation technique can help to essentially improve the convergence conditions of the successive approximations for solutions of systems of non-linear ordinary differential equations under non-local boundary conditions. The application of the technique is shown on an example of a problem with non-linear integral boundary conditions involving values of the unknown function and its derivative.

Keywords: non-local boundary conditions, parametrisation, successive approximations, interval division.

2010 Mathematics Subject Classification: 34B15.

1 Introduction

Recently, boundary value problems with non-local conditions for non-linear differential equations have attracted much attention (see, e. g., the editorial note [1] and the rest of the issue for extensive references). Problems with non-local boundary conditions are usually treated by using equivalent reformulation as a suitable fixed point or coincidence equation, for which purpose, as a rule, one uses Green's operator of a linearised problem. The process of approximation of the solution based directly on this kind of representations, however, may be quite complicated.

A reasonably efficient way to deal with this kind of problems is provided by methods of numerical-analytic type (see, e. g., [3]). Since convergence conditions often involve terms proportional to the length of the time interval, the conditions needed for the applicability of this type of methods can be significantly weakened if one constructs the scheme using a

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suitable interval division. It turns out that, by introducing a single intermediate point, one can weaken the convergence conditions by half at the cost of one more variable in the parameter list (see [5–7]). In [7], where mainly the periodic problem is considered, we also note that it is possible to consider multiple interval divisions. A scheme of this kind, which is applicable to the case of general boundary conditions, is constructed in the present note.

The approach that we are going to discuss is based on a suitable parametrisation, so that the values of approximations to a solution are monitored at multiple time instants. In this way, it can be regarded as an efficient alternative to the multiple shooting [2, 10] and may be well applicable also in the cases where shooting procedures fail. The latter may happen either because of the complicated character of the boundary conditions (according to our knowledge, the currently available shooting schemes are designed for the cases where the boundary conditions are local two-point) or, more importantly, due to the failure to satisfy the basic assumptions needed to apply the method. Indeed, one may note that shooting methods require the existence of sufficiently many derivatives of the non-linearity (in particular, because Newton-like methods are commonly used to solve the corresponding numerical equations, see, e.g., [10, p. 516] or [11, p. 375]). Furthermore, in order to carry out shooting, one has to be sure that the initial value problem for the differential equation in question has always a unique solution defined on the entire given time interval. The smoothness of the non-linearity alone is insufficient: consider, e. g., $u' = u^2$ on $[a, b]$ with $u(a) = 1/(\lambda - a)$, where $a < \lambda < b$; then the solution $u(t) = 1/(\lambda - t)$ is undefined at $t = \lambda$. Last, but not least, the existence of a solution is usually assumed *a priori* when applying shooting methods. In contrast to this, the approach that we suggest here, in many cases, allows one to prove the solvability of the problem in a rigorous way (see, e. g., [7, 9]).

Here, we study the non-linear boundary value problem

$$u'(t) = f\,(t, u(t))\,, \qquad t \in [a, b]\,, \tag{1.1}$$

$$\phi(u) = d, \tag{1.2}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $d \in \mathbb{R}^n$ is a given vector, and ϕ is a vector functional on the space of absolutely continuous functions (generally speaking, non-linear).

Following the idea used in numerical methods for approximate solution of initial value problems for ordinary differential equations, let us fix a natural N and choose $N + 1$ grid points

$$t_0 = a, \qquad t_k = t_{k-1} + h_k, \quad k = 1, \dots, N-1, \qquad t_N = b, \tag{1.3}$$

where h_k, $k = 1, \dots, N-1$, are the corresponding step sizes. Thus, $[a, b]$ is divided into N subintervals $[t_0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$, ..., $[t_{N-1}, t_N]$. Of course, one can use a constant step size in (1.3): $h_k = N^{-1}(b - a)$, $k = 1, 2, \dots, N$; the more general form (1.3), however, may allow one to pose better conditions on the non-linearity in the corresponding region.

The aim of this note is to present an approach to problems of type (1.1)–(1.2) which is similar in principle to [7] and is also based on reductions to certain simpler problems with unknown parameters. The auxiliary two-point problems are constructed here with multiple interval divisions, which leads one to convergence conditions significantly weaker than in the case of a single intermediate point. Here, in contrast to the case of linear two-point conditions discussed in [6, 7], the exact fulfilment of the boundary condition for approximations is not guaranteed any more (of course, the boundary condition is satisfied exactly in the limit). The advantage is, however, that many different types of boundary conditions can be thus handled in a unified way, the specific properties of the problem being transferred to the determining equations. It seems that, in the case of general boundary value problems, interval division for approximations constructed analytically is employed here for the first time.

2 Notation

We fix an $n \in \mathbb{N}$ and a bounded closed set $D \subset \mathbb{R}^n$. For vectors $x = \operatorname{col}(x_1, \dots, x_n) \in \mathbb{R}^n$ the obvious notation $|x| = \operatorname{col}(|x_1|, \dots, |x_n|)$ is used and the inequalities between vectors are understood componentwise. The same convention is adopted implicitly for operations like "max" and "min".

$\mathbf{1}_n$ and $\mathbf{0}_n$ are, respectively, the unit and zero matrices of dimension n.

$r(K)$ is the maximal, in modulus, eigenvalue of a matrix K.

For a set $D \subset \mathbb{R}^n$, closed interval $[a, b] \subset \mathbb{R}$, Carathéodory function $f : [a, b] \times D \to \mathbb{R}^n$, $n \times n$ matrix K with non-negative entries, we write$f \in \operatorname{Lip}_K(D)$ if the inequality

$$|f(t, u) - f(t, v)| \le K |u - v| \tag{2.1}$$

holds for all $\{u, v\} \subset D$ and a.e. $t \in [a, b]$.

If $\varrho \in \mathbb{R}^n$ is a non-negative vector, by the componentwise ϱ-neighbourhood of a point $z \in \mathbb{R}^n$ we understand the set

$$O_\varrho(z) := \{\xi \in \mathbb{R}^n : |\xi - z| \le \varrho\}. \tag{2.2}$$

Similarly, the componentwise ϱ-neighbourhood of a set $\Omega \subset \mathbb{R}^n$ is defined as

$$O_\varrho(\Omega) := \bigcup_{\xi \in \Omega} O_\varrho(\xi). \tag{2.3}$$

For given two bounded connected sets $D_0 \subset \mathbb{R}^n$ and $D_1 \subset \mathbb{R}^n$, introduce the set

$$\mathcal{B}(D_0, D_1) := \{(1 - \theta)\xi + \theta\eta : \xi \in D_0,\ \eta \in D_1, \theta \in [0, 1]\}. \tag{2.4}$$

Finally, given a set $D \subset \mathbb{R}^n$ and a function $f : [a, b] \times D \to \mathbb{R}^n$, we put

$$\delta_{[\tau_1,\tau_2],D}(f) := \operatorname*{ess\,sup}_{(t,x) \in [\tau_1,\tau_2] \times D} f(t, x) - \operatorname*{ess\,inf}_{(t,x) \in [\tau_1,\tau_2] \times D} f(t, x) \tag{2.5}$$

for any $\{\tau_1, \tau_2\} \subset [a, b]$, $\tau_1 < \tau_2$.

The sequence of functions $\alpha_m(\cdot, \tau, l) : [\tau, \tau + l] \to [0, \infty)$, $m = 0, 1, \dots$, where $l \in (0, \infty)$, is defined by the relations

$$\alpha_0(t, \tau, l) := 1, \tag{2.6}$$

$$\alpha_{m+1}(t, \tau, l) := \left(1 - \frac{t - \tau}{l}\right) \int_\tau^t \alpha_m(s, \tau, l)\, ds + \frac{t - \tau}{l} \int_t^{\tau + l} \alpha_m(s, \tau, l)\, ds \tag{2.7}$$

for all $t \in [\tau, \tau + l]$ and $m \ge 0$. Functions (2.7) have the following properties essentially used below.

Lemma 2.1 ([3, Lemma 3.16]). *Let τ and l be given. Then, for all $t \in [\tau, \tau + l]$, the functions $\alpha_m(\cdot, \tau, l)$, $m \ge 1$, satisfy the estimates*

$$\alpha_{m+1}(t, \tau, l) \le \frac{10}{9} \left(\frac{3l}{10}\right)^m \alpha_1(t, \tau, l) \tag{2.8}$$

if $m \geq 0$ *and*

$$\alpha_{m+1}(t,\tau,l) \leq \frac{3l}{10}\alpha_m(t,\tau,l) \tag{2.9}$$

if $m \geq 2$.

Lemma 2.2 ([4, Lemma 2])**.** *For an arbitrary essentially bounded function* $f : [\tau, \tau + l] \to \mathbb{R}^n$*, the estimate*

$$\left| \int_\tau^t \left(f(\tau) - \frac{1}{l} \int_\tau^{\tau+l} f(s)\, \mathrm{d}s \right) \mathrm{d}\tau \right| \leq \frac{1}{2}\alpha_1(t,\tau,l) \left(\operatorname*{ess\,sup}_{s\in[\tau,\tau+l]} f(s) - \operatorname*{ess\,inf}_{s\in[\tau,\tau+l]} f(s) \right) \tag{2.10}$$

is true for a.e. $t \in [\tau, \tau + l]$.

It follows from (2.7) that

$$\alpha_1(t,\tau,l) = 2\,(t-\tau)\left(1 - \frac{t-\tau}{l}\right), \qquad t \in [\tau, \tau+l], \tag{2.11}$$

and $\max_{t\in[\tau,\tau+l]} \alpha_1(t,\tau,l) = l/2$.

3 Parametrisation and auxiliary problems

3.1 Parameter sets

Let us fix certain closed bounded sets

$$D_k \subset \mathbb{R}^n, \qquad k = 0, 1, 2, \ldots, N, \tag{3.1}$$

and focus on the absolutely continuous solutions u of problem (1.1)–(1.2) whose values at nodes (1.3) lie in the corresponding sets (3.1), i. e., the solutions u such that

$$u(t_k) \in D_k, \qquad k = 0, 1, 2, \ldots, N. \tag{3.2}$$

Given sets (3.1), we introduce the sets

$$D_{k-1,k} := \mathcal{B}(D_{k-1}, D_k), \qquad k = 1, 2, \ldots, N, \tag{3.3}$$

and, for any non-negative vector ϱ, put

$$\Omega_k(\varrho) := O_\varrho(D_{k-1,k}), \qquad k = 1, 2, \ldots, N. \tag{3.4}$$

Recall that, according to (2.3), (2.4), $D_{k-1,k}$ is the set of all possible straight line segments joining points of D_{k-1} with points of D_k, whereas $\Omega_k(\varrho)$ is the componentwise ϱ-neighbourhood of $D_{k-1,k}$.

3.2 Freezing

The idea that we are going to use suggests to replace the original non-local problem (1.1)–(1.2) by a suitable family of model boundary value problems with simpler boundary conditions (see, e. g., [8,9]. Let us do this in the following way. Consider the vectors

$$z^{(k)} = \operatorname{col}(z_1^{(k)}, z_2^{(k)}, \dots, z_n^{(k)}), \qquad k = 0, 1, 2, \dots, N, \tag{3.5}$$

where N is the number of nodes from (1.3). These vectors will be regarded as unknown parameters whose values are to be determined. Let us "freeze" the values of u at the nodes (1.3) by formally putting

$$u(t_k) = z^{(k)}, \qquad k = 0, 1, 2, \dots, N, \tag{3.6}$$

and consider the restrictions of equation (1.1) to each of the subintervals of the division:

$$x'(t) = f(t, x(t)), \qquad t \in [t_{k-1}, t_k]. \tag{3.7}$$

Then, in a natural way, we have

$$x(t_{k-1}) = z^{(k-1)}, \qquad x(t_k) = z^{(k)}, \quad k = 1, 2, \dots, N. \tag{3.8}$$

For any fixed $k = 1, 2, \dots, N$, relations (3.7), (3.8) can be regarded formally as an overdetermined boundary value problem with two-point boundary conditions containing unknown parameters $z^{(k-1)}$ and $z^{(k)}$. This leads one to a kind of reduction principle where, instead of the original equation (1.1), one considers the parametrised problems (3.7), (3.8) and tries to determine the appropriate value of $z^{(0)}$, $z^{(1)}$, ..., $z^{(N)}$.

Due to the form of the boundary condition (3.8), it is natural to apply to (3.7), (3.8) techniques similar to those used in [7] for two-point problems. This is done in Section 4.2 below, where the successive approximations $x_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)})$, $m \geq 0$, defined, respectively, on the intervals

$$[t_{k-1}, t_k], \qquad k = 1, 2, \dots, N, \tag{3.9}$$

are constructed. Note that the differential equation (3.7) is considered on an interval of length h_k (see (1.3)).

4 Interval division and successive approximations

4.1 Assumptions

Let us fix the sets $D_k \subset \mathbb{R}^n$, $k = 0, 1, \dots, N$, from (3.1). We make the following assumptions.

Assumption 4.1. There exist non-negative vectors $\varrho^{(1)}$, $\varrho^{(2)}$, ..., $\varrho^{(N)}$ such that

$$\varrho^{(k)} \geq \frac{h_k}{4} \delta_{[t_{k-1}, t_k], \Omega_k(\varrho^{(k)})}(f) \tag{4.1}$$

for all $k = 1, 2, \dots, N$.

Recall that $\Omega_k(\varrho^{(k)})$ is a $\varrho^{(k)}$-neighbourhood of $D_{k-1,k}$ (see (3.4)). We suppose that f is Lipschitzian, in the space variable, on the sets $\Omega_k(\varrho^{(k)})$, $k = 1, 2, \dots, N$. Namely,

Assumption 4.2. There exist non-negative matrices K_1, K_2, ..., K_N such that

$$f \in \operatorname{Lip}_{K_k}(\Omega_k(\varrho^{(k)})), \qquad k = 1, 2, \dots, N. \tag{4.2}$$

Finally, we assume in the sequel that the matrices K_1, K_2, ..., K_N involved in (4.2) satisfy the conditions

$$r(K_k) < \frac{10}{3h_k}, \qquad k = 1, \dots, N. \tag{4.3}$$

Assumptions 4.1 and 4.2, together with condition (4.3), are used to prove the applicability of the techniques described below. They mean essentially that the non-linearities in the equation are Lipschitzian on sufficiently large domains ($\varrho^{(k)}$ satisfies inequality (4.1)) with sufficiently small constants (condition (4.3)). It should be noted, however, that (4.1) and (4.3) are both satisfied if the number N of nodes in (1.3) is large enough. Thus, the basic and, in fact, the only restrictive assumption in this note is that f is Lipschitzian on a bounded set.

4.2 Successive approximations

For any fixed values $z^{(0)}, z^{(1)}, \dots, z^{(N)}$, define the sequences of functions $x_m^{(k)} : [t_{k-1}, t_k] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, N$, $m = 0, 1, 2, \dots$, by putting

$$x_0^{(k)}\left(t, z^{(k-1)}, z^{(k)}\right) := \left(1 - \frac{t - t_{k-1}}{h_k}\right) z^{(k-1)} + \frac{t - t_{k-1}}{h_k} z^{(k)}, \tag{4.4}$$

$$\begin{aligned} x_m^{(k)}\left(t, z^{(k-1)}, z^{(k)}\right) := z^{(k-1)} &+ \int_{t_{k-1}}^{t} f\left(s, x_{m-1}^{(k)}\left(s, z^{(k-1)}, z^{(k)}\right)\right) \mathrm{d}s \\ &- \frac{t - t_{k-1}}{h_k} \int_{t_{k-1}}^{t_k} f\left(s, x_{m-1}^{(k)}\left(s, z^{(k-1)}, z^{(k)}\right)\right) \mathrm{d}s \\ &+ \frac{t - t_{k-1}}{h_k} \left(z^{(k)} - z^{(k-1)}\right) \end{aligned} \tag{4.5}$$

for all $m = 1, 2, \dots$ and $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N$.

In view of (4.4), relation (4.5) can be represented alternatively as

$$\begin{aligned} x_m^{(k)}\left(t, z^{(k-1)}, z^{(k)}\right) = x_0^{(k)}\left(t, z^{(k-1)}, z^{(k)}\right) &+ \int_{t_{k-1}}^{t} f\left(s, x_{m-1}^{(k)}\left(s, z^{(k-1)}, z^{(k)}\right)\right) \mathrm{d}s \\ &- \frac{t - t_{k-1}}{h_k} \int_{t_{k-1}}^{t_k} f\left(s, x_{m-1}^{(k)}\left(s, z^{(k-1)}, z^{(k)}\right)\right) \mathrm{d}s. \end{aligned} \tag{4.6}$$

One can see from (4.4) that the graphs of the functions $x_0^{(k)}\left(\cdot, z^{(k-1)}, z^{(k)}\right)$, $k = 1, 2, \dots, N$, form a broken line joining the points $(t_k, z^{(k)})$, $k = 1, 2, \dots, N$. By virtue of (4.6), this implies, in particular, that all the functions (4.5) have property (3.8), i. e.,

$$x_m^{(k)}\left(t_{k-1}, z^{(k-1)}, z^{(k)}\right) = z^{(k-1)}, \qquad x_m^{(k)}\left(t_k, z^{(k-1)}, z^{(k)}\right) = z^{(k)} \tag{4.7}$$

for any $k = 1, 2, \dots, N$, independently of the values of $z^{(0)}, z^{(1)}, \dots, z^{(N)}$.

5 Convergence of successive approximations

It turns out that the sequences $\{x_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)}) : m \geq 0\}$, $k = 1, 2, \dots, N$ given by (4.4) and (4.5) are helpful for the investigation of solutions of the given problem (1.1)–(1.2).

Theorem 5.1. *Let Assumptions 4.1 and 4.2 hold and, moreover, the corresponding matrices $K_1, K_2, \dots, K_N$ satisfy condition (4.3). Then, for any $(z^{(0)}, z^{(1)}, \dots, z^{(N)}) \in D_0 \times D_1 \times \cdots \times D_N$ and $k = 1, 2, \dots, N$:*

1. *The limit*
$$\lim_{m\to\infty} x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) =: x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)}) \tag{5.1}$$
exists uniformly in $t \in [t_{k-1}, t_k]$.

2. *The limit function* (5.1) *satisfies the conditions*
$$x_\infty^{(k)}(t_{k-1}, z^{(k-1)}, z^{(k)}) = z^{(k-1)}, \qquad x_\infty^{(k)}(t_k, z^{(k-1)}, z^{(k)}) = z^{(k)}. \tag{5.2}$$

3. *The function* $x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)})$ *is the unique absolutely continuous solution of the integral equation*
$$x(t) = z^{(k-1)} + \int_{t_{k-1}}^{t} f(s, x(s))\,ds - \frac{t - t_{k-1}}{h_k}\int_{t_{k-1}}^{t_k} f(s, x(s))\,ds \\ + \frac{t - t_{k-1}}{h_k}(z^{(k)} - z^{(k-1)}), \qquad t \in [t_{k-1}, t_k]. \tag{5.3}$$

4. *The estimate*
$$\left| x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)}) - x_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)}) \right| \le \frac{5}{9}\alpha_1(t, t_{k-1}, h_k) R_{m,k}\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \tag{5.4}$$
holds for $m \ge 0$, $t \in [t_{k-1}, t_k]$, *where*
$$R_{m,k} := \left(\frac{3}{10}h_k K_k\right)^m \left(\mathbf{1}_n - \frac{3}{10}h_k K_k\right)^{-1}. \tag{5.5}$$

Proof. The proof is carried out similarly to that of [8, Theorem 3]. Let us fix arbitrary vectors $z^{(i)} \in D_i$, $i = 0, 1, \dots, N$, and a number $k \in \{1, 2, \dots, N\}$. We first show that, under the conditions assumed,
$$\left\{ x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) : (t, z^{(k-1)}, z^{(k)}) \in [t_{k-1}, t_k] \times D_{k-1} \times D_k \right\} \subset \Omega_k(\varrho^{(k)}) \tag{5.6}$$
for any $m \ge 0$. Indeed, the validity of (5.6) for $m = 0$ is an immediate consequence of (4.4). Let us put
$$r_m^{(k)}(t, \xi, \eta) = |x_m^{(k)}(t, \xi, \eta) - x_{m-1}^{(k)}(t, \xi, \eta)|, \tag{5.7}$$
where $m = 1, 2, \dots$, $(\xi, \eta) \in D_{k-1} \times D_k$. Due to estimate (2.10) of Lemma 2.2 with $\tau = t_{k-1}$, $l = h_k$, relations (4.4) and (4.5) yield
$$\begin{aligned} r_1^{(k)}(t, z^{(k-1)}, z^{(k)}) &\le \frac{1}{2}\alpha_1(t, t_{k-1}, h_k)\Bigg(\operatorname*{ess\,sup}_{t\in[t_{k-1},t_k]} f(t, x_0^{(k)}(t, z^{(k-1)}, z^{(k)})) \\ &\qquad - \operatorname*{ess\,inf}_{t\in[t_{k-1},t_k]} f(t, x_0^{(k)}(t, z^{(k-1)}, z^{(k)}))\Bigg) \\ &\le \frac{1}{2}\alpha_1(t, t_{k-1}, h_k)\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \\ &\le \frac{h_k}{4}\delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \end{aligned} \tag{5.8}$$

for all $t \in [t_{k-1}, t_k]$, $(\xi, \eta) \in D_{k-1} \times D_k$. In view of (4.1), this means that $x_1^{(k)}(t, \xi, \eta) \in \Omega_k(\varrho^{(k)})$ whenever $(t, \xi, \eta) \in [t_{k-1}, t_k] \times D_{k-1} \times D_k$, i.e., (5.6) holds for $m = 1$. Using this and arguing by induction with the help of Lemma 2.2, we easily establish that

$$\begin{aligned} |x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) - x_0^{(k)}(t, z^{(k-1)}, z^{(k)})| &\leq \frac{1}{2}\alpha_1(t, t_{k-1}, h_k)\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \\ &\leq \frac{h_k}{4}\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \end{aligned} \tag{5.9}$$

for $k = 1, 2, \ldots, N$ and $m \geq 2$. Therefore, (5.6) holds for any $m \geq 0$.

In view of (4.4), (4.5), the identity

$$\begin{aligned} &x_{m+1}^{(k)}(t, z^{(k-1)}, z^{(k)}) - x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) \\ &\quad = \int_{t_{k-1}}^{t} \left[f(s, x_m^{(k)}(s, z^{(k-1)}, z^{(k)})) - f(s, x_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)})) \right] \mathrm{d}s \\ &\qquad - \frac{t - t_{k-1}}{h_k} \int_{t_{k-1}}^{t_k} \left[f(s, x_m^{(k)}(s, z^{(k-1)}, z^{(k)})) - f(s, x_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)})) \right] \mathrm{d}s \end{aligned} \tag{5.10}$$

holds. Using equality (5.10), Assumption 4.2 and Lemmata 2.1 and 2.2, we obtain

$$\begin{aligned} r_2^{(k)}(t, z^{(k-1)}, z^{(k)}) &\leq \frac{1}{2} K_k \left(\left(1 - \frac{t - t_{k-1}}{h_k}\right) \int_{t_{k-1}}^{t} \alpha_1(s, t_{k-1}, h_k)\, \mathrm{d}s \right. \\ &\qquad \left. + \frac{t - t_{k-1}}{h_k} \int_{t}^{t_k} \alpha_1(s, t_{k-1}, h_k)\, \mathrm{d}s \right) \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \\ &\leq \frac{1}{2} K_k\, \alpha_2(t, t_{k-1}, h_k)\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \\ &\leq \frac{5}{9} \left(\frac{3}{10} h_k K_k \right) \alpha_1(t, t_{k-1}, h_k)\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \end{aligned} \tag{5.11}$$

for $t \in [t_{k-1}, t_k]$. One then easily shows by induction that

$$\begin{aligned} r_{m+1}^{(k)}(t, z^{(k-1)}, z^{(k)}) &\leq K_k^m \alpha_{m+1}(t, t_{k-1}, h_k)\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \\ &\leq \frac{5}{9} \left(\frac{3}{10} h_k K_k \right)^m \alpha_1(t, t_{k-1}, h_k)\, \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \end{aligned} \tag{5.12}$$

for $t \in [t_{k-1}, t_k]$. Therefore, in view of (5.12)

$$\begin{aligned} \left| x_{m+j}^{(k)}(t, z^{(k-1)}, z^{(k)}) - x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) \right| &\leq \sum_{i=1}^{j} r_{m+i}^{(k)}(t, z^{(k-1)}, z^{(k)}) \\ &\leq \frac{5}{9} \alpha_1(t, t_{k-1}, h_k) \sum_{i=1}^{j} \left(\frac{3}{10} h_k K_k \right)^{m+i-1} \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \\ &= \frac{5}{9} \alpha_1(t, t_{k-1}, h_k) \left(\frac{3}{10} h_k K_k \right)^m \\ &\quad \times \sum_{i=0}^{j-1} \left(\frac{3}{10} h_k K_k \right)^i \delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f) \end{aligned} \tag{5.13}$$

for all $m \geq 0$ and $j \geq 1$. Recall that $\delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f)$ is computed according to (2.5). Since, due to (4.3), $r(\frac{3}{10} h_k K_k) < 1$, we have $\lim_{m\to\infty} (\frac{3}{10} h_k K_k)^m = \mathbf{0}_n$ and $\sum_{i=0}^{j-1} (\frac{3}{10} h_k K_k)^i \leq (\mathbf{1}_n - \frac{3}{10} h_k K_k)^{-1}$

for any j. Therefore, (5.13) and the Cauchy criterion imply the existence of a uniform limit in (5.1). Equalities (5.2) are an immediate consequence of (4.7). Finally, passing to the limit as $m \to \infty$ in (4.5) and (5.13), we show that the limit function satisfies (5.3) and obtain estimate (5.4). It remains to recall the arbitrariness of $z^{(0)}, z^{(1)}, \ldots, z^{(N)}$ and k. $\square$

Theorem 5.1 implies, in particular, that one can introduce the function $\Delta^{(k)} : D^{k-1} \times D^k \to \mathbb{R}^n$ by putting

$$\Delta^{(k)}(\xi, \eta) := \eta - \xi - \int_{t_{k-1}}^{t_k} f(s, x_\infty^{(k)}(s, \xi, \eta))\, ds \tag{5.14}$$

for all $(\xi, \eta) \in D_{k-1} \times D_k$, $k = 1, 2, \ldots, N$. Then it follows immediately from (5.3) that the following statement holds.

Corollary 5.2. *Let the conditions of Theorem 5.1 hold. Let $z^{(j)} \in D_j$, $j = 0, 1, \ldots, N$, be arbitrary. Then, for any $k = 1, 2, \ldots, N$, the function $x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)}) : [t_{k-1}, t_k]$ is the solution of the Cauchy problem*

$$x'(t) = f(t, x(t)) + \frac{1}{h_k}\Delta^{(k)}(z^{(k-1)}, z^{(k)}), \qquad t \in [t_{k-1}, t_k], \tag{5.15}$$

$$x(t_{k-1}) = z^{(k-1)}, \tag{5.16}$$

where $\Delta^{(k)} : D^{k-1} \times D^k \to \mathbb{R}^n$ is given by (5.14).

Note that, by (2.11), $\alpha_1(t, t_{k-1}, h_k) \le h_k/2$ and, therefore, (5.4) implies the estimate

$$\left|x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)}) - x_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)})\right| \le \frac{5h_k}{18} R_{m,k}\, \delta_{[t_{k-1}, t_k], \Omega_k(\varrho^{(k)})}(f) \tag{5.17}$$

for any $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, N$, with $R_{m,k}$ given by (5.5).

6 Limit functions and determining equations

It is natural to expect that the limit functions $x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)}) : [t_{k-1}, t_k] \to \mathbb{R}^n$, $k = 0, 1, \ldots, N$, of iterations (4.5) may help one to state general criteria of solvability of problem (1.1), (1.2). Such criteria can be formulated in terms of the respective functions $\Delta^{(k)} : D_{k-1} \times D_k \to \mathbb{R}^n$, $k = 0, 1, \ldots, N$, given by equalities (5.14) that provide such a conclusion. Indeed, Theorem 5.1 ensures that, under the conditions assumed, the functions $x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)}) : [t_{k-1}, t_k] \to \mathbb{R}^n$, $k = 1, 2, \ldots, N$, are well defined for all $(z^{(k-1)}, z^{(k)}) \in D_{k-1} \times D_k$. Therefore, by putting

$$u_\infty(t, z^{(0)}, z^{(1)}, \ldots, z^{(N)}) := \begin{cases} x_\infty^{(1)}(t, z^{(0)}, z^{(1)}) & \text{if } t \in [t_0, t_1], \\ x_\infty^{(2)}(t, z^{(1)}, z^{(2)}) & \text{if } t \in [t_1, t_2], \\ \quad\vdots & \\ x_\infty^{(N)}(t, z^{(N-1)}, z^{(N)}) & \text{if } t \in [t_{N-1}, t_N], \end{cases} \tag{6.1}$$

we obtain a function $u_\infty(\cdot, z^{(0)}, z^{(1)}, \ldots, z^{(N)}) : [a, b] \to \mathbb{R}^n$, which is well defined for all the values $z^{(k)} \in D_k$, $k = 0, 1, \ldots, N$. This function is obviously continuous because, at the points $t = t_k$, we have

$$x_\infty^{(k)}(t_k, z^{(k-1)}, z^{(k)}) = x_\infty^{(k+1)}(t_k, z^{(k)}, z^{(k+1)}) \tag{6.2}$$

for all $k = 1, 2, \ldots, N-1$. Equalities (6.2) follow immediately from the fact that the function $x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)})$ is a solution of equation (5.3).

The following theorem establishes a relation of function (6.1) to the solution of the boundary value problem (1.1)–(1.2) in terms of the zeroes of the functions $\Delta^{(k)}$, $k = 1, 2, \ldots, N$.

Theorem 6.1. *Let the conditions of Theorem 5.1 hold. Then:*

1. *The function $u_\infty(\cdot, z^{(k-1)}, z^{(k)}) : [a, b] \to \mathbb{R}^n$ defined by (6.1) is an absolutely continuous solution of problem (1.1)–(1.2) if and only if the vectors $z^{(k)}$, $k = 0, 1, \ldots, N$, satisfy the system of $n(N+1)$ numerical equations*

$$\begin{gathered} \Delta^{(k)}(z^{(k-1)}, z^{(k)}) = 0, \qquad k = 1, 2, \ldots, N, \\ \Delta^{(N+1)}(z^{(0)}, z^{(1)}, \ldots, z^{(N)}) = 0, \end{gathered} \tag{6.3}$$

where $\Delta^{(N+1)} : D_0 \times D_1 \times \cdots \times D_N \to \mathbb{R}^n$ is defined as

$$(z^{(0)}, z^{(1)}, \ldots, z^{(N)}) \longmapsto \Delta^{(N+1)}(z^{(0)}, z^{(1)}, \ldots, z^{(N)}) := \phi(u_\infty(\cdot, z^{(0)}, z^{(1)}, \ldots, z^{(N)})) - d.$$

2. *For every solution $u(\cdot)$ of problem (1.1)–(1.2) with $u(t_k) \in D_k$, $k = 0, 1, \ldots, N$, there exist vectors $z^{(k)}$, $k = 0, 1, \ldots, N$, such that*

$$u(\cdot) = u_\infty(\cdot, z^{(0)}, z^{(1)}, \ldots, z^{(N)}). \tag{6.4}$$

This statement is proved similarly to [5, Theorem 4]. Equations (6.3) are usually referred to as *determining* or *bifurcation* equations because their roots determine solutions of the original problem.

7 Approximate determining equations

Although Theorem 6.1 provides a complete theoretical answer to the question on the construction of a solution of problem (1.1)–(1.2), its application faces complications since it is difficult to find the limit function (5.1) and, as a consequence, the functions $\Delta^{(k)} : D_{k-1} \times D_k \to \mathbb{R}^n$, $k = 1, 2, \ldots, N$, and $\Delta^{(N+1)} : D_0 \times D_1 \times \cdots \times D_N \to \mathbb{R}^n$, appearing in (6.3) are usually unknown explicitly. The complication can be overcome if we replace the unknown limit $x_\infty^{(k)}(\cdot, z^{(k-1)}, z^{(k)})$ by an iteration $x_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)})$ of form (4.5) for a fixed m and put

$$u_m(t, z^{(0)}, z^{(1)}, \ldots, z^{(N)}) := \begin{cases} x_m^{(1)}(t, z^{(0)}, z^{(1)}) & \text{if } t \in [t_0, t_1], \\ x_m^{(2)}(t, z^{(1)}, z^{(2)}) & \text{if } t \in [t_1, t_2], \\ \qquad \vdots & \\ x_m^{(N)}(t, z^{(N-1)}, z^{(N)}) & \text{if } t \in [t_{N-1}, t_N]. \end{cases} \tag{7.1}$$

We see that (7.1) is an approximate version of the unknown function (6.1). Its values can be found explicitly for all $t \in [a, b]$ and $z^{(k)} \in D_k$, $k = 0, 1, 2, \ldots, N$.

Considering function (7.1), we arrive in a natural way to the so-called *approximate determining equations:*

$$\begin{gathered} \Delta_m^{(k)}(z^{(k-1)}, z^{(k)}) = 0, \qquad k = 1, 2, \ldots, N, \\ \Delta^{(N+1)}(z^{(0)}, z^{(1)}, \ldots, z^{(N)}) = 0, \end{gathered} \tag{7.2}$$

where

$$\Delta_m^{(k)}(z^{(k-1)}, z^{(k)}) := z^{(k)} - z^{(k-1)} - \int_{t_{k-1}}^{t_k} f(s, x_m^{(k)}(s, z^{(k-1)}, z^{(k)}))\,\mathrm{d}s, \qquad k = 1, 2, \ldots, N,$$
$$\Delta_m^{(N+1)}(z^{(0)}, z^{(1)}, \ldots, z^{(N)}) := \phi(u_m(\cdot, z^{(0)}, z^{(1)}, \ldots, z^{(N)})) - d.$$

Note that, unlike system (6.3), the mth approximate determining system (7.2) contains only terms involving the functions $x_m^{(j)}(\cdot, z^{(j-1)}, z^{(j)})$, $j = 1, 2, \ldots, N$, and, thus, known explicitly.

Let $(\tilde{z}^{(0)}, \tilde{z}^{(1)}, \ldots, \tilde{z}^{(N)})$ be a solution of the approximate determining system (7.2) for a certain value of m. Then the function

$$[a, b] \ni t \longmapsto U_m(t) := u_m(t, \tilde{z}^{(0)}, \tilde{z}^{(1)}, \ldots, \tilde{z}^{(N)})$$

is natural to be regarded as the mth approximation to a solution of the boundary value problem (1.1)–(1.2). In particular, it follows from (5.17) that

$$\left|x_\infty^{(k)}(\cdot, \tilde{z}^{(k-1)}, \tilde{z}^{(k)}) - U_m(t)\right| \le \frac{5h_k}{18}\left(\frac{3}{10}h_k K_k\right)^m \left(1_n - \frac{3}{10}h_k K_k\right)^{-1} \delta_{[t_{k-1}, t_k], \Omega_k(\varrho^{(k)})}(f) \tag{7.3}$$

for any $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, N$.

The existence of a solution can be analysed based on the approximate determining equations (7.2) similarly to [3, 9], this topic is not considered here. In relation to estimate (7.3) one may note that, according to Theorem 6.1, the solution necessarily has form (6.4) with certain values of $z^{(k)}$, $k = 0, 1, \ldots, N$. Thus, we have $z^{(k)} \approx \tilde{z}^{(k)}$, $k = 0, 1, \ldots, N$, and, therefore, $x_\infty^{(k)}(t, \tilde{z}^{(k-1)}, \tilde{z}^{(k)})$ is an approximation of $x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)})$, which is the value of the exact solution for $t \in [t_{k-1}, t_k]$.

8 Example

Let us demonstrate the approach described above on a model example. Consider the system of differential equations

$$\begin{aligned} x_1'(t) &= \frac{1}{2}(x_2(t))^2 - \frac{t}{4}x_1(t) + \frac{t^2(t-1)}{32} + \frac{9t}{40}, \\ x_2'(t) &= \frac{t}{8}x_1(t) - t^2 x_2(t) + \frac{15}{64}t^3 + \frac{t}{80} + \frac{1}{4}, \qquad t \in [0, 1.9], \end{aligned} \tag{8.1}$$

with the integral boundary conditions

$$\begin{aligned} \int_0^{1.9} \left(s x_1(s) x_2(s) + \frac{1}{4}x_1'(s)\right) \mathrm{d}s &= \frac{10099697}{48000000}, \\ \int_0^{1.9} \left(s^2 x_2^2(s) + \frac{1}{4}x_2'(s)\right) \mathrm{d}s &= \frac{3426099}{8000000}. \end{aligned} \tag{8.2}$$

Clearly, problem (8.1), (8.2) is a particular case of (1.1)–(1.2) with $a = 0$, $b = 1.9$, $d \approx \operatorname{col}(0.21, 0.428)$, $x \mapsto \phi(x) := \operatorname{col}\left(\int_0^{1.9}(s x_1(s) x_2(s) + \frac{1}{4}x_1'(s))\,\mathrm{d}s, \int_0^{1.9}(s^2 x_2^2(s) + \frac{1}{4}x_2'(s))\,\mathrm{d}s\right)$, $(x_1, x_2) \mapsto f(t, x_1, x_2) := \operatorname{col}\left(\frac{1}{2}x_2^2 - \frac{1}{4}t x_1 + \frac{1}{32}t^2(t-1) + \frac{9}{40}t, \frac{1}{8}t x_1 - t^2 x_2 + \frac{15}{64}t^3 + \frac{1}{80}t + \frac{1}{4}\right)$. It is easy to check that

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{10}, \qquad x_2^*(t) = \frac{t}{4} \tag{8.3}$$

$x_1^*(0)$	$x_2^*(0)$	$x_1^*(1)$	$x_2^*(1)$	$x_1^*(1.5)$	$x_2^*(1.5)$	$x_1^*(1.9)$	$x_2^*(1.9)$
−0.1	0	0.025	0.25	0.18125	0.375	0.35125	0.475

Table 8.1: Values of functions (8.3) at nodes (8.4).

is a solution of problem (8.1), (8.2).

Let us choose the grid (1.3) with $N = 3$ and the nodes

$$t_0 := 0, \qquad t_1 := 1, \qquad t_2 := 1.5, \qquad t_3 := 1.9. \tag{8.4}$$

Then, obviously,

$$h_1 = 1, \qquad h_2 = \frac{1}{2}, \qquad h_3 = \frac{2}{5}. \tag{8.5}$$

According to (3.5), the scheme will depend on four two-dimensional vector parameters $z^{(k)}$, $0 \le k \le 3$; their meaning is explained by Table 8.2.

Variable	$z^{(0)}$	$z^{(1)}$	$z^{(2)}$	$z^{(3)}$
Value it approximates	$x(0)$	$x(1)$	$x(1.5)$	$x(1.9)$

Table 8.2: The meaning of the parameters in the example.

The number of the solutions of the algebraic determining system (7.2) coincides with the number of the solutions of the given problem. Different solutions have to be detected by changing appropriately the initial domains D_k, $0 \le k \le 3$. Let us carry out several steps of iteration with two different choices of the initial domains and the radii of neighbourhoods.

8.1 First solution

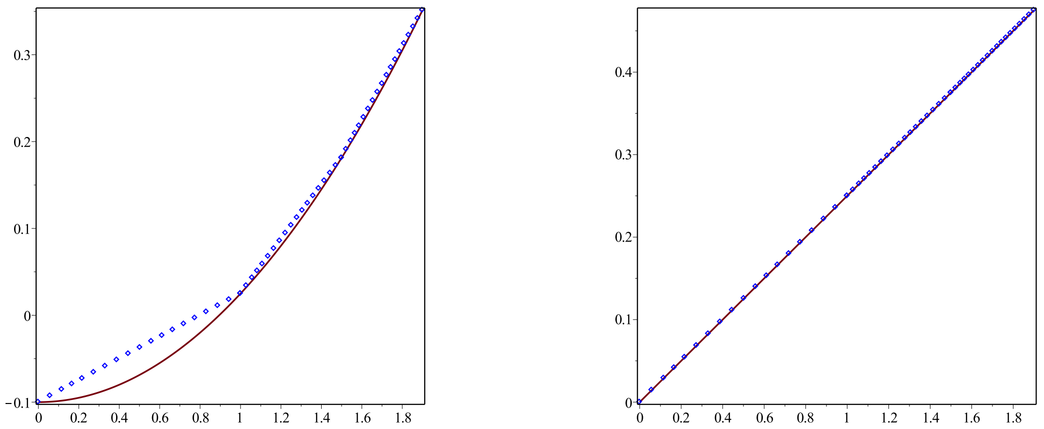
Let us choose the sets D_k, $0 \le k \le 3$, as follows:

$$\begin{aligned} D_0 &:= \{(x_1, x_2) : -0.15 \le x_1 \le 0.182,\ -0.01 \le x_2 \le 0.38\}, \\ D_1 &:= \{(x_1, x_2) : 0.024 \le x_1 \le 0.182,\ 0.24 \le x_2 \le 0.38\}, \\ D_2 &:= D_1, \\ D_3 &:= \{(x_1, x_2) : 0.024 \le x_1 \le 0.352,\ 0.24 \le x_2 \le 0.48\}. \end{aligned} \tag{8.6}$$

This choice can be justified by the fact that the zeroth approximate determining system (i.e., (7.2) with $m = 0$) has roots lying in these sets (see the first column in Table 8.3). Furthermore, sets (8.6) contain the corresponding parts of the graph of the zeroth approximation. The graphs of the components of the latter function, which, according to (4.4), have the form of broken lines, are shown on Figure 8.1.

Using (3.3), we find that the corresponding sets $D_{k-1,k}$, $0 \le k \le 3$, have the form

$$\begin{aligned} D_{0,1} &= \{(x_1, x_2) : -0.15 \le x_1 \le 0.182,\ -0.01 \le x_2 \le 0.38\}, \\ D_{1,2} &= \{(x_1, x_2) : 0.024 \le x_1 \le 0.182,\ 0.24 \le x_2 \le 0.38\}, \\ D_{2,3} &= \{(x_1, x_2) : 0.024 \le x_1 \le 0.352,\ 0.24 \le x_2 \le 0.48\}. \end{aligned}$$



(a) First component (b) Second component

Figure 8.1: The zeroth approximation to solution (8.3).

In order to construct suitable sets on which Assumptions 4.1 and 4.2 will be verified, we need to choose vectors $\varrho^{(1)}$, $0 \leq k \leq 3$. Let us put, for example,

$$\varrho^{(1)} := \operatorname{col}(0.2, 0.3), \qquad \varrho^{(2)} := \operatorname{col}(0.1, 0.2), \qquad \varrho^{(3)} := \operatorname{col}(0.1, 0.4). \tag{8.7}$$

Then, according to formula (3.4), the corresponding sets $\Omega_k(\varrho^{(1)})$, $0 \leq k \leq 3$, have the form

$$\begin{aligned} \Omega_1(\varrho^{(1)}) &= \{(x_1, x_2) : -0.35 \leq x_1 \leq 0.382, \ -0.31 \leq x_2 \leq 0.68\}, \\ \Omega_2(\varrho^{(2)}) &= \{(x_1, x_2) : -0.076 \leq x_1 \leq 0.282, \ 0.04 \leq x_2 \leq 0.58\}, \\ \Omega_3(\varrho^{(3)}) &= \{(x_1, x_2) : -0.076 \leq x_1 \leq 0.452, \ -0.16 \leq x_2 \leq 0.88\}. \end{aligned} \tag{8.8}$$

	$m = 0$	$m = 1$	$m = 2$	$m = 9$
$z_1^{(0)}$	-0.1035005019	-0.09996763819	-0.09999692457	-0.1000000003
$z_2^{(0)}$	-0.001518357199	-0.00005070186245	$-6.201478977 \cdot 10^{-6}$	$2.976222204 \cdot 10^{-10}$
$z_1^{(1)}$	0.01941727634	0.02499481248	0.02500348592	0.02499999977
$z_2^{(1)}$	0.2496837722	0.2499856345	0.2499933349	0.2500000002
$z_1^{(2)}$	0.1756874698	0.1812419151	0.1812534461	0.1812499999
$z_2^{(2)}$	0.3748370539	0.3749950964	0.3749933900	0.3750000000
$z_1^{(3)}$	0.3748370539	0.3512417410	0.3512532909	0.3512500000
$z_2^{(3)}$	0.4748456880	0.4749990066	0.4749934809	0.4750000000

Table 8.3: Approximate values of the parameters for the first solution on several steps of approximation.

A direct computation shows that the Lipschitz condition (4.2) for the right-hand side terms of (8.1) holds in $\Omega_1(\varrho^{(1)})$, $\Omega_2(\varrho^{(2)})$, $\Omega_3(\varrho^{(3)})$, respectively, with the matrices

$$K_1 = \begin{pmatrix} 1/4 & 17/25 \\ 1/8 & 1 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 3/8 & 29/50 \\ 3/16 & 9/4 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 19/40 & 22/25 \\ 19/80 & 361/100 \end{pmatrix}. \tag{8.9}$$

Then, by (8.5), we obtain

$$\begin{aligned} r(K_1) &= 1.1 < \frac{10}{3} = \frac{10}{3h_1}, \\ r(K_2) &= \frac{21}{16} + \frac{7}{80}\sqrt{129} \approx 2.3063 < 6.6667 \approx \frac{20}{3} = \frac{10}{3h_2}, \\ r(K_3) &= \frac{817 + \sqrt{426569}}{400} \approx 3.6753 < 8.3334 \approx \frac{25}{3} = \frac{10}{3h_3}. \end{aligned} \tag{8.10}$$

Relations (8.10) show that matrices (8.9) satisfy conditions (4.3) with the step sizes (8.5). Furthermore, in view of (8.5), (8.7), and (8.8), we have

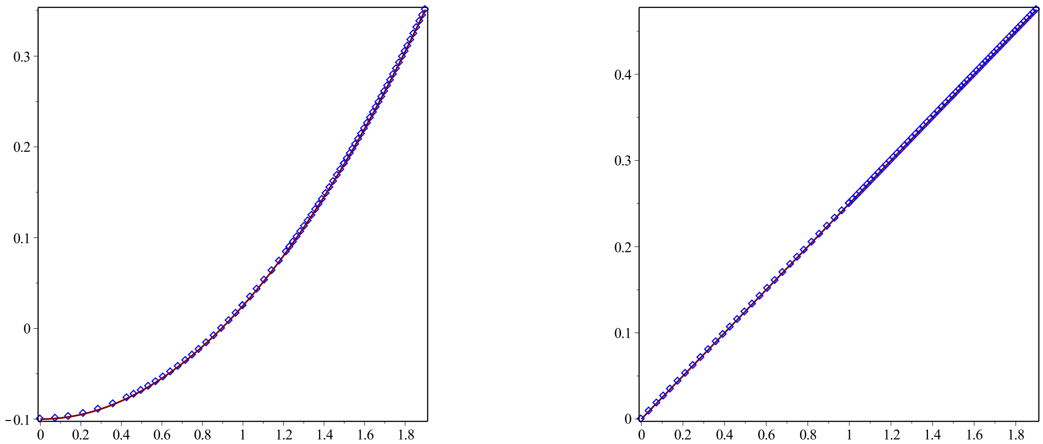
$$\frac{h_1}{4}\delta_{[t_0,t_1],\Omega_1(\varrho^{(1)})}(f) = \frac{1}{4}\delta_{[0,1],\Omega_1(\varrho^{(1)})}(f) = \frac{1}{4}\begin{pmatrix}0.5437\\1.0815\end{pmatrix} = \begin{pmatrix}0.135925\\0.270375\end{pmatrix} \leq \begin{pmatrix}0.2\\0.3\end{pmatrix} = \varrho^{(1)},$$

$$\frac{h_2}{4}\delta_{[t_1,t_2],\Omega_2(\varrho^{(2)})}(f) = \frac{1}{8}\delta_{[1,1.5],\Omega_2(\varrho^{(2)})}(f) = \frac{1}{8}\begin{pmatrix}0.41405625\\1.282125\end{pmatrix} \approx \begin{pmatrix}0.05175703125\\0.160265625\end{pmatrix} \leq \begin{pmatrix}0.1\\0.2\end{pmatrix} = \varrho^{(2)},$$

$$\frac{h_3}{4}\delta_{[t_2,t_3],\Omega_3(\varrho^{(3)})}(f) = \frac{1}{10}\delta_{[1.5,1.9],\Omega_3(\varrho^{(3)})}(f) = \frac{1}{10}\begin{pmatrix}0.753925\\3.882175\end{pmatrix} = \begin{pmatrix}0.0753925\\0.3882175\end{pmatrix} \leq \begin{pmatrix}0.1\\0.4\end{pmatrix} = \varrho^{(3)},$$

which means that vectors (8.7) satisfy conditions (4.1) of Assumption 4.1.

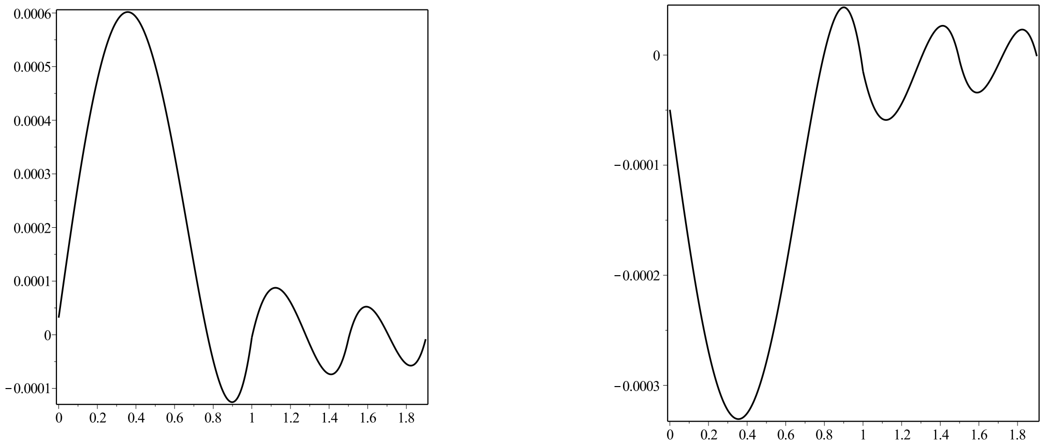
Thus, we see that that all the conditions of Theorem 5.1 are fulfilled, and the sequences of functions (4.5) for this example are convergent. Using *Maple 14* for constructing the iterations and solving the approximate determining equations (7.2) for $m = 0, 1, 2, 9$, we obtain the numerical results shown in Table 8.3.



(a) First component (b) Second component

Figure 8.2: The first solution ((8.3), solid line) and its first approximation (dots).

We may note at this point that, at nodes (8.4), the pair of functions (8.3), which, as has been indicated, is a solution of problem (8.1), (8.2), has the values listed in Table 8.1. Comparing Tables 8.3 and 8.1, we find enough evidence to claim that the results of computation with the present choice of initial domains correspond to solution (8.3). This is further confirmed when we put the components of this function and the first approximation ($m = 1$) on the same plot (see Figure 8.2). The graphs of higher approximations (we have carried out computations up to $m = 9$) practically coincide with one another and there is no way to distinguish them in the given resolution.



(a) First component (b) Second component

Figure 8.3: Error of the first approximation to solution (8.3).

Considering the difference between the approximation and solution (8.3), e. g., for $m = 1$, we see that the maximal error is about $6 \cdot 10^{-4}$ (see Figure 8.3). All this, together with Tables 8.3 and 8.1, demonstrates a rather high quality of approximation.

8.2 Second solution

Let us now check the results of computation with a different choice of the sets D_k, $0 \leq k \leq 3$. Instead of (8.6), we put

$$
\begin{aligned}
D_0 &:= \{(x_1, x_2) : -0.3 \leq x_1 \leq 0.11,\ -0.65 \leq x_2 \leq 0.16\},\\
D_1 &:= \{(x_1, x_2) : -0.05 \leq x_1 \leq 0.11,\ -0.22 \leq x_2 \leq 0.16\},\\
D_2 &:= D_1,\\
D_3 &:= \{(x_1, x_2) : -0.05 \leq x_1 \leq 0.27,\ -0.22 \leq x_2 \leq 0.404\}.
\end{aligned}
$$

According to (3.3), the corresponding sets $D_{0,1}$, $D_{1,2}$ and $D_{2,3}$ have the form

$$
\begin{aligned}
D_{0,1} &= \{(x_1, x_2) : -0.3 \leq x_1 \leq 0.11,\ -0.65 \leq x_2 \leq 0.16\},\\
D_{1,2} &= \{(x_1, x_2) : -0.05 \leq x_1 \leq 0.11,\ -0.22 \leq x_2 \leq 0.16\},\\
D_{2,3} &= \{(x_1, x_2) : -0.05 \leq x_1 \leq 0.27,\ -0.22 \leq x_2 \leq 0.404\}.
\end{aligned}
$$

Putting now

$$
\varrho^{(1)} = \operatorname{col}(0.3, 0.6), \qquad \varrho^{(2)} = \operatorname{col}(0.1, 0.3), \qquad \varrho^{(3)} = \operatorname{col}(0.15, 0.9), \tag{8.11}
$$

we find from formula (3.4) that, in this case,

$$
\begin{aligned}
\Omega_1(\varrho^{(1)}) &= \{(x_1, x_2) : -0.6 \leq x_1 \leq 0.41,\ -1.25 \leq x_2 \leq 0.76\},\\
\Omega_2(\varrho^{(2)}) &= \{(x_1, x_2) : -0.15 \leq x_1 \leq 0.21,\ -0.52 \leq x_2 \leq 0.46\},\\
\Omega_3(\varrho^{(3)}) &= \{(x_1, x_2) : -0.2 \leq x_1 \leq 0.42,\ -1.12 \leq x_2 \leq 1.304\}.
\end{aligned} \tag{8.12}
$$

We see that sets (8.8) and (8.12) essentially differ from one another.

	$m=0$	$m=1$	$m=2$	$m=7$	$m=9$
$z_1^{(0)}$	-0.3130351578	-0.2915938662	-0.2899053146	-0.2913487961	-0.2913488037
$z_2^{(0)}$	-0.6853367388	-0.6463772306	-0.6463171293	-0.6452155574	-0.6452156078
$z_1^{(1)}$	-0.0657496383	-0.0462927056	-0.0440494448	-0.0456699676	-0.0456699553
$z_2^{(1)}$	-0.2581141824	-0.2183359461	-0.2180852446	-0.2170072781	-0.2170073139
$z_1^{(2)}$	0.0838541806	0.1001891237	0.1021532831	0.1006981868	0.1006981978
$z_2^{(2)}$	0.1398968489	0.1589542884	0.1589307544	0.1594479818	0.1594479635
$z_1^{(3)}$	0.2497893002	0.2659558438	0.2675903580	0.2664281996	0.2664282026
$z_2^{(3)}$	0.4027863216	0.4033132063	0.4036275942	0.4033665823	0.4033666220

Table 8.4: Approximate values of the parameters for the second solution.

Using (8.5), (8.11), (8.12) and computing the corresponding values $\delta_{[t_{k-1},t_k],\Omega_k(\varrho^{(k)})}(f)$, $0 \le k \le 3$, we get

$$\frac{h_1}{4}\delta_{[t_0,t_1],\Omega_1(\varrho^{(1)})}(f) = \frac{1}{4}\delta_{[0,1],\Omega_1(\varrho^{(1)})}(f) = \frac{1}{4}\begin{pmatrix}1.15625\\2.13625\end{pmatrix} = \begin{pmatrix}0.2890625\\0.5340625\end{pmatrix} \le \begin{pmatrix}0.3\\0.6\end{pmatrix} = \varrho^{(1)},$$
$$\frac{h_2}{4}\delta_{[t_1,t_2],\Omega_2(\varrho^{(2)})}(f) = \frac{1}{8}\delta_{[1,1.5],\Omega_2(\varrho^{(2)})}(f) \approx \frac{1}{8}\begin{pmatrix}0.39160625\\2.289850344\end{pmatrix} \approx \begin{pmatrix}0.04895\\0.28623\end{pmatrix} \le \begin{pmatrix}0.1\\0.3\end{pmatrix} = \varrho^{(2)},$$
$$\frac{h_3}{4}\delta_{[t_2,t_3],\Omega_3(\varrho^{(3)})}(f) = \frac{1}{10}\delta_{[1.5,1.9],\Omega_3(\varrho^{(3)})}(f) = \frac{1}{10}\begin{pmatrix}1.259083\\8.89789\end{pmatrix} = \begin{pmatrix}0.1259083\\0.889789\end{pmatrix} \le \begin{pmatrix}0.15\\0.9\end{pmatrix} = \varrho^{(3)}.$$

The last estimates imply that (4.1) holds for vectors (8.11) and, therefore, Assumption 4.1 is satisfied. A further computation shows that the Lipschitz condition (4.2) holds on the respective sets (8.12) with the matrices

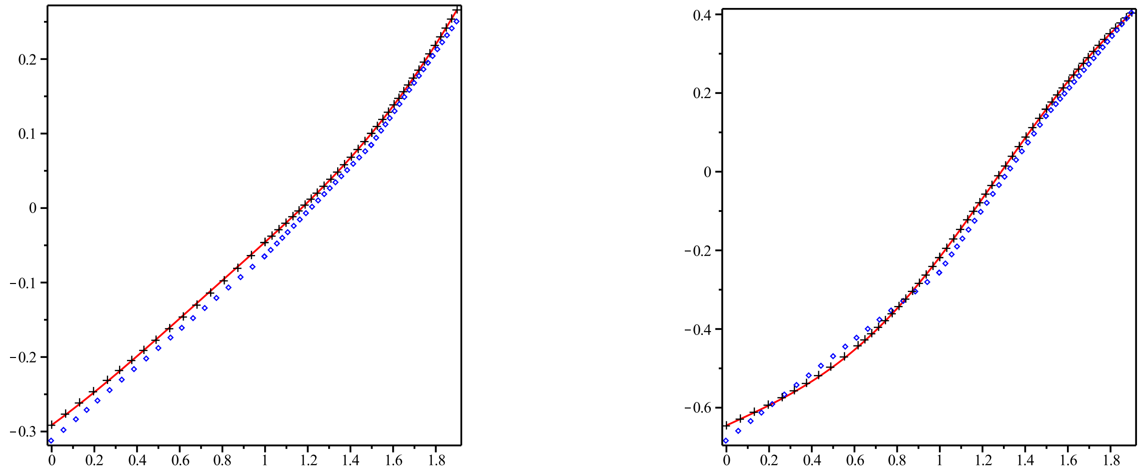
$$K_1 = \begin{pmatrix}1/4 & 19/25\\1/8 & 1\end{pmatrix}, \qquad K_2 = \begin{pmatrix}3/8 & 19/25\\3/16 & 9/4\end{pmatrix}, \qquad K_3 = \begin{pmatrix}19/40 & 163/125\\19/80 & 361/100\end{pmatrix}, \tag{8.13}$$

for which one finds that

$$\begin{aligned} r(K_1) &= \frac{5}{8} + \frac{\sqrt{377}}{40} \approx 1.1104 < \frac{10}{3} = \frac{10}{3h_1},\\ r(K_2) &= \frac{21}{16} + \frac{\sqrt{6537}}{80} \approx 2.3231 < 6.6667 \approx \frac{20}{3} = \frac{10}{3h_2},\\ r(K_3) &= \frac{817 + \sqrt{442681}}{400} \approx 3.7059 < 8.3334 \approx \frac{25}{3} = \frac{10}{3h_3}. \end{aligned} \tag{8.14}$$

It follows from relations (8.14) that matrices (8.13) satisfy conditions (4.3) with h_1, h_3, and h_3 given by (8.5).

Carrying out computations, we see that the approximate determining systems (7.2), along with the solution found in Section 8.1 in sets (8.8) (see Table 8.3), has another solution in sets (8.12). The corresponding approximate values of parameters at several steps of iteration ($m = 0, 1, 2, 7, 9$) are presented in Table 8.4. In particular, we see that, as in Section 8.1, the piecewise linear zeroth approximation provides a useful hint as to where the solution should be looked for (see the first column of Table 8.4).



(a) First component (b) Second component

Figure 8.4: The zeroth (⋄), first (+), and ninth (solid line) approximations to the second solution.

The graphs of three approximations to this solution ($m = 0, 1, 9$) are shown on Figure 8.4. The residual obtained as a result of substitution of the ninth approximation into the given differential system (8.1) is of order of 10^{-8}.

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A constructive approach to boundary value problems with state-dependent impulses

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ABSTRACT

We investigate the non-linear system of ordinary differential equations

$$u'(t) = f(t, u(t)), \quad \text{a.e. } t \in [a, b],$$

subject to the state-dependent impulse condition

$$u(t+) - u(t-) = \gamma(u(t-)) \quad \text{for } t \in (a, b) \text{ such that } g(t, u(t-)) = 0$$

and the linear two-point boundary condition

$$Au(a) + Cu(b) = d.$$

Here, $-\infty < a < b < \infty$, f and γ are given continuous vector-functions, g is a continuous scalar function, A, C are constant matrices, and d is a constant vector. The instants of time t where the jump occurs are determined by the equation $g(t, u(t-)) = 0$ and, thus, are unknown *a priori* and essentially depend on the solution u. We discuss a reduction technique allowing one to combine the analysis of existence of solutions with an efficient construction of approximate solutions. At present, according to the authors' knowledge, no numerical results for boundary value problems with state-dependent impulses are available in the literature.

1. Introduction

We consider the non-linear system of differential equations

$$u'(t) = f(t, u(t)) \qquad \text{a. e. } t \in [a, b], \tag{1}$$

with $-\infty < a < b < \infty$ and a continuous $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$. Eq. (1) is subject to the *state-dependent* impulse condition

$$u(t+) - u(t-) = \gamma(u(t-)) \quad \text{for } t \text{ such that } g(t, u(t-)) = 0. \tag{2}$$

Here $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ are continuous, and the impulse instants $t \in (a, b)$ in (2) are unknown. These instants are called state-dependent because they depend on the solution u itself through the equation $g(t, u(t-)) = 0$.

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The impulsive problem (1), (2) is investigated together with the linear boundary condition

$$Au(a) + Cu(b) = d, \tag{3}$$

where d is a constant vector, and A, C are constant matrices, satisfying the condition

$$\operatorname{rank}[A, C] = n,$$

where the brackets indicate the formation of the appropriate block $n \times 2n$ matrix. The non-singularity of these matrices is not required.

In the context of systems (1) with impulse action of type (2), the set

$$G = \{(t, x) \in [a, b] \times \mathbb{R}^n : g(t, x) = 0\} \tag{4}$$

determined by the function g from (2) is usually called a *barrier*. We study here those solutions of the system which are allowed to meet the barrier finitely many times only (and, thus, exclude the beating phenomenon from consideration). More precisely, the following definition is adopted.

Definition 1.1. A left-continuous vector-function $u : [a, b] \to \mathbb{R}^n$ is called a *solution* of problem (1)–(3) if (3) holds and there exist an integer p and certain points $\tau_i \in (a, b)$, $i = 1, \ldots, p$, such that $a < \tau_1 < \tau_2 < \cdots < \tau_p < b$, the restrictions $u|_{[a,\tau_1]}$, $u|_{(\tau_1,\tau_2]}, \ldots,$ $u|_{(\tau_p,b]}$ have continuous derivatives and

1. u satisfies (1) for $t \in [a, b] \setminus \{\tau_1, \tau_2, \ldots, \tau_p\}$;
2. $u(\tau_i+) - u(\tau_i) = \gamma(u(\tau_i))$ and $g(\tau_i, u(\tau_i)) = 0$ for $i = 1, \ldots, p$.

We see that, for any $i = 1, \ldots, p$, the trajectory of u intersects the barrier G at the time τ_i (i.e., $(\tau_i, u(\tau_i)) \in G$) and has a jump of size $\gamma(u(\tau_i))$. It should be noted that both the time instants $\tau_1, \ldots, \tau_p$ and the very value of p, generally speaking, depend on u, so that different solutions may have jumps at different points.

For classical monographs about impulsive problems, see [1–3]. Studies of real life problems with state-dependent impulsive effects can be found in [4–9]. Many papers are devoted to state-dependent impulsive initial value problems, where the existence, stability and other asymptotic properties of solutions are studied (e. g. [10–17]). Optimal control problems for rather general classes of systems with state-dependent jumps are investigated in [18–20]. For the studies of state-dependent impulsive periodic problems, we can refer, e. g., to [21–26]. As regards other types of state-dependent impulsive boundary value problems, one cannot name but a few papers dealing with them (see [27–33]) since the majority of results on boundary value problems for impulsive systems concern jumps at fixed times. This is due to the fact that state-dependent impulses significantly change properties of boundary value problems, which is explained in detail in [32]. Last, but not least, we did not succeed in finding any kind of numerical results for state-dependent impulsive boundary value problems, which would provide one a way to find approximations to a solution. This, along with the above-said, contributed to our motivation to study problem (1)–(3). We shall show here that there is a relatively simple way to approach this problem from a constructive point of view.

In this initial study, we focus our attention at the case where $p = 1$ for any solution u under consideration (i. e., u meets the barrier G exactly once) and use the techniques which have been applied in [34] in a different situation and allow us to examine the solvability of problem (1)–(3) as well as to find approximate solutions. Our approach is based on the construction of two simple model problems (namely, (17)–(19) and (37)–(39) in Section 3) depending on parameters τ, ξ, λ, and η. Under certain additional conditions one shows that, for all values of parameters from suitable bounded sets, solutions of the auxiliary problems can be obtained as limits of uniformly convergent successive approximations (15), (16) and (35), (36). Equations in the parameterized model problems contain functional perturbation terms which essentially depend on the parameters and which, together with the original boundary conditions (3) and the barrier crossing condition (4), produce a system of finitely many *determining* equations (more precisely, system of equations (55)). Numerical values of the parameters should be found from (55) in the bounded sets mentioned above. A solution of problem (1)–(3) is then constructed (see (42)) by means of solutions of those problems (17)–(19) and (37)–(39) which are determined by the values of parameters satisfying (55). Consequently, the infinite-dimensional problem (1)–(3) is reduced to the finite-dimensional system (55).

In practice, we study system (55) through its approximate versions, where the unknown limit function is replaced by certain successive approximations that are computed explicitly (see (61) in Section 5). This allows one to formulate conditions sufficient for the solvability of (55) and we get approximate solutions of problem (1)–(3) and error estimates using computer algebra systems (for example, Maple 14, which has been applied in our present work). According to our best knowledge, this is the first numerical-analytic method for this type of impulsive problems. It can be applied for problems with either linear or non-linear boundary conditions, which is shown for problems without impulses in [34–40]. Furthermore, we can work in this way with barriers described in the implicit form (4), in contrast to the most frequently considered case where the barrier is given as

$$\{(t, x) : t = g(x)\}. \tag{5}$$

In particular, the results of [29–33] are applicable to the case of (5) only.

The example of a boundary value problem with a state-dependent jump given in Section 6 shows that the method can also be applied to obtain certain multiplicity results.

2. Notation and auxiliary inequalities

In the sequel, for any vector $x = \operatorname{col}(x_1, \dots, x_n) \in \mathbb{R}^n$ the obvious notation $|x| = \operatorname{col}(|x_1|, \dots, |x_n|)$ is used and inequalities between vectors are understood componentwise. A similar convention is adopted implicitly in the case of the operations 'max' and 'min' for vector-valued functions. The symbols 1_n and 0_n stand respectively for the unit and zero matrix of dimension n, and $r(K)$ denotes the maximal, in modulus, eigenvalue of a square matrix K.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a set and $\varrho \in \mathbb{R}^n$ be a vector with non-negative components. By the *componentwise* ϱ*-neighborhood* of Ω, we mean the set

$$\mathcal{O}_\varrho(\Omega) := \bigcup_{\xi \in \Omega} B(\xi, \varrho), \tag{6}$$

where

$$B(z, \varrho) := \{\xi \in \mathbb{R}^n : |\xi - z| \le \varrho\}.$$

Definition 2.2. Given two sets Ω_0 and Ω_1 in $\mathbb{R}^n$, we put

$$\mathcal{B}(\Omega_0, \Omega_1) := \{\theta\xi + (1-\theta)\eta : \xi \in \Omega_0, \eta \in \Omega_1, \theta \in [0, 1]\}. \tag{7}$$

The set $\mathcal{B}(\Omega_0, \Omega_1)$ is thus formed by all possible straight line segments joining points of Ω_0 with those from Ω_1. It is obvious from (7) that $\mathcal{B}(\Omega_0, \Omega_1) \subset \operatorname{conv}(\Omega_0 \cup \Omega_1)$ but the equality is, generally speaking, not true.

In the sequel, we write $f \in \operatorname{Lip}_K(\Omega)$ if $\Omega \subset \mathbb{R}^n$ is a compact set, K is a non-negative square matrix of dimension n, and a function $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the componentwise Lipschitz condition in the space variable

$$\left| f(t, u_1) - f(t, u_2) \right| \le K|u_1 - u_2| \tag{8}$$

for all u_1, u_2 from Ω and $t \in [a, b]$. We also use the notation

$$\delta_\Omega(f) := \max_{(t,\xi)\in[a,b]\times\Omega} f(t, \xi) - \min_{(t,\xi)\in[a,b]\times\Omega} f(t, \xi). \tag{9}$$

If $a \le t_0 < t_1 \le b$, we put $\alpha_0(t; t_0, t_1) := 1$ and

$$\alpha_m(t; t_0, t_1) = \left(1 - \frac{t - t_0}{t_1 - t_0}\right) \int_{t_0}^{t} \alpha_{m-1}(s; t_0, t_1)\, ds + \frac{t - t_0}{t_1 - t_0} \int_{t}^{t_1} \alpha_{m-1}(s; t_0, t_1)\, ds \tag{10}$$

for $t \in [t_0, t_1]$ and $m = 1, 2, \dots$. In particular, (10) implies that

$$\alpha_1(t; a, b) = 2(t - a)\left(1 - \frac{t - a}{b - a}\right), \qquad t \in [a, b], \tag{11}$$

and

$$\max_{t\in[a,b]} \alpha_1(t; a, b) = (b - a)/2.$$

Lemma 2.3 [37, Lemma 3.16]**.** *Let $a \le t_0 < t_1 \le b$. Then the estimates*

$$\alpha_m(t; t_0, t_1) \le \frac{10}{9}\left(\frac{3(t_1 - t_0)}{10}\right)^{m-1} \alpha_1(t; t_0, t_1) \tag{12}$$

hold for all $t \in [t_0, t_1]$ and $m = 1, 2, \dots$.

Lemma 2.4 [37, Lemma 3.13]**.** *Let $a \le t_0 < t_1 \le b$ and $y : [t_0, t_1] \to \mathbb{R}^n$ be a continuous function. Then*

$$\left| \int_{t_0}^{t} \left(y(\sigma) - \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} y(s)\, ds \right) d\sigma \right| \le \frac{1}{2}\alpha_1(t; t_0, t_1) \left(\max_{s\in[t_0,t_1]} y(s) - \min_{s\in[t_0,t_1]} y(s) \right) \tag{13}$$

for $t \in [t_0, t_1]$.

3. Parameterized iterations

In the sequel, we adopt the following convention: if Ω is a set in $\mathbb{R}^n$ and certain other related sets are denoted by Ω_s with some expressions s, then Ω_{t+} stands for the set $\left\{x + \gamma(x) : x \in \Omega_{t-}\right\}$ for any $t \in (a, b)$. Keeping this notation in mind, let us fix an arbitrary point $\tau \in (a, b)$, choose certain compact convex sets Ω_a, $\Omega_{\tau-}$, and Ω_b in $\mathbb{R}^n$, and introduce the set

$$\Omega_{\tau+} := \left\{x + \gamma(x) : x \in \Omega_{\tau-}\right\}.$$

In this way, the set $\Omega_{\tau+}$ is obtained from $\Omega_{\tau-}$ by shifting the latter using the jump map γ from (2). According to (7), we construct the sets

$$\Omega_{a,\tau-} := \mathcal{B}(\Omega_a, \Omega_{\tau-}), \quad \Omega_{\tau+,b} := \mathcal{B}(\Omega_{\tau+}, \Omega_b). \tag{14}$$

The idea that we are going to follow is to attempt to approximate the solution of (1), (2) by suitable sequences of functions *separately* for the times preceding the unknown moment of jump and those corresponding to the after-jump evolution, whereas the jump time itself is treated as a parameter to be determined later. It turns out that this can be done in a manner resembling the construction of iterations for two-point boundary value problems [40]. More precisely, consider a scalar parameter $\tau \in (a, b)$ together with three vector parameters $\xi \in \Omega_a$, $\lambda \in \Omega_{\tau-}$, and $\eta \in \Omega_b$. The key role in our analysis will be played by the values τ and λ representing, respectively, the unknown jump time of the solution and its after-jump value.

3.1. Iterations for the pre-jump evolution

Let us start by introducing the parameterized sequence of functions $\{x_m(\cdot; \tau, \xi, \lambda) : m \geq 0\}$ according to the relations

$$x_0(t; \tau, \xi, \lambda) := \left(1 - \frac{t-a}{\tau - a}\right)\xi + \frac{t-a}{\tau-a}\lambda, \tag{15}$$

$$x_m(t; \tau, \xi, \lambda) := x_0(t; \tau, \xi, \lambda) + \int_a^t f(s, x_{m-1}(s; \tau, \xi, \lambda))\, ds - \frac{t-a}{\tau-a}\int_a^\tau f(s, x_{m-1}(s; \tau, \xi, \lambda))\, ds \tag{16}$$

for all $t \in [a, \tau]$ and $m \geq 1$. These formulas are motivated by the following simple proposition.

Proposition 3.1. *Let $\tau \in (a, b)$, $\xi \in \Omega_a$, and $\lambda \in \Omega_{\tau-}$ be fixed. Then*

$$x_m(a; \tau, \xi, \lambda) = \xi$$

and

$$x_m(\tau; \tau, \xi, \lambda) = \lambda$$

for any $m \geq 0$. Furthermore, if $\lim_{m\to\infty} x_m(\cdot; \tau, \xi, \lambda) =: x(\cdot)$ exists uniformly on $[a, \tau]$, then $x(\cdot)$ is a solution of the problem

$$x'(t) = f(t, x(t)) + \frac{1}{\tau - a}\left(\lambda - \xi - \int_a^\tau f(s, x(s))\, ds\right), \qquad t \in [a, \tau], \tag{17}$$

$$x(a) = \xi, \tag{18}$$

$$x(\tau) = \lambda. \tag{19}$$

By a *solution* of the two-point problem (17)–(19), we mean a continuously differentiable function x with properties (19) and (18) satisfying (17) at every point of $[a, \tau]$.

Proof. The indicated properties of $x_m(\cdot; \tau, \xi, \lambda)$, $m \geq 0$, are direct consequences of (15) and (16). Therefore, passing to the limit as $m \to \infty$ in equality (16), we find that $x(\cdot)$ satisfies the integral equation

$$x(t) = \xi + \int_a^t f(s, x(s))\, ds - \frac{t-a}{\tau-a}\int_a^\tau f(s, x(s))\, ds + \frac{t-a}{\tau-a}(\lambda - \xi), \qquad t \in [a, \tau], \tag{20}$$

whence (17)–(19) follow immediately. □

The following statement establishes the uniform convergence of sequence (15), (16).

Theorem 3.2. *Let there exist a non-negative vector ϱ with the property*

$$\varrho \geq \frac{b-a}{4}\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f) \tag{21}$$

such that $f \in \mathrm{Lip}_K(\mathcal{O}_\varrho(\Omega_{a,\tau-}))$ with a certain matrix K. If, moreover, K satisfies the condition

$$r(K) < \frac{10}{3(b-a)}, \tag{22}$$

then, for any $\tau \in (a, b)$, $\xi \in \Omega_a$, and $\lambda \in \Omega_{\tau-}$:

1. *Functions (16) are continuously differentiable on $[a, \tau]$ for any $m \geq 0$;*
2. *$\{x_m(t; \tau, \xi, \lambda) : t \in [a, \tau],\ m \geq 0\} \subset \mathcal{O}_\varrho(\Omega_{a,\tau-})$;*
3. *$\{x_m(\cdot; \tau, \xi, \lambda) : m \geq 0\}$ converges to a limit function $x_\infty(\cdot; \tau, \xi, \lambda)$ uniformly on $[a, \tau]$;*
4. *$x_\infty(\cdot; \tau, \xi, \lambda)$ is a solution of the boundary value problem (17)–(19) and this problem has no other solutions with values in $\mathcal{O}_\varrho(\Omega_{a,\tau-})$;*

5. *The estimate*

$$\left|x_\infty(t;\tau,\xi,\lambda)-x_m(t;\tau,\xi,\lambda)\right|\le\frac{5}{9}\alpha_1(t;a,\tau)\,K_*^m(1_n-K_*)^{-1}\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f) \tag{23}$$

holds for all $t\in[a,\tau]$ *and* $m\ge1$, *where*

$$K_*:=\frac{3}{10}(b-a)K. \tag{24}$$

Proof. We can argue by induction similarly to [34]. Assume that $\tau\in(a,b)$, $\xi\in\Omega_a$, and $\lambda\in\Omega_{\tau-}$ are fixed and ϱ satisfies (21). We start by showing that, under the conditions assumed, the values of $x_m(\cdot;\tau,\xi,\lambda)$ do not escape from the ϱ-neighborhood of the set $\Omega_{a,\tau-}$ for any $m\ge0$.

Indeed, it is obvious from (15) that

$$\{x_0(t;\tau,\xi,\lambda):t\in[a,\tau]\}\subset\Omega_{a,\tau-}$$

because, for all $t\in[a,\tau]$, the value $x_0(t;\tau,\xi,\eta)$ is a convex combination of ξ and η and, thus, belongs to $\mathcal{B}(\Omega_a,\Omega_{\tau-})$. Hence, the inclusion

$$\{x_m(t;\tau,\xi,\lambda):t\in[a,\tau]\}\subset\mathcal{O}_\varrho(\Omega_{a,\tau-}) \tag{25}$$

is true for $m=0$. Assume that (25) holds for a certain $m\ge1$. Then, using relations (16), (25), (11) and Lemma 2.4 with $t_0=a$ and $t_1=\tau$, we obtain

$$\begin{aligned}\left|x_{m+1}(t;\tau,\xi,\lambda)-x_0(t;\tau,\xi,\lambda)\right|&\le\frac{1}{2}\alpha_1(t;a,\tau)\left(\max_{s\in[a,\tau]}f(s,x_m(s;\tau,\xi,\lambda))-\min_{s\in[a,\tau]}f(s,x_m(s;\tau,\xi,\lambda))\right)\\&\le\frac{1}{2}\alpha_1(t;a,\tau)\,\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f)\end{aligned} \tag{26}$$

$$\le\frac{b-a}{4}\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f) \tag{27}$$

for all $t\in[a,\tau]$. Recall that we use notation (9). In view of (25) with $m=0$ and (21), it follows from (27) that all the values of $x_m(\cdot;\tau,\xi,\lambda)$ lie in the ϱ-neighborhood of $\Omega_{a,\tau-}$. This means that (25) remains true when m is incremented to $m+1$ and, thus, holds for any $m\ge0$.

Using now (15), (16) and introducing the notation

$$r_m(t;\tau,\xi,\lambda):=\left|x_m(t;\tau,\xi,\lambda)-x_{m-1}(t;\tau,\xi,\lambda)\right| \tag{28}$$

for $t\in[a,\tau]$, $m\ge1$, we obtain

$$\begin{aligned}r_m(t;\tau,\xi,\lambda)=\Bigg|&\left(1-\frac{t-a}{\tau-a}\right)\int_a^t(f(s,x_m(s;\tau,\xi,\lambda))-f(s,x_{m-1}(s;\tau,\xi,\lambda)))\,\mathrm{d}s\\&-\frac{t-a}{\tau-a}\int_t^\tau(f(s,x_m(s;\tau,\xi,\lambda))-f(s,x_{m-1}(s;\tau,\xi,\lambda)))\,\mathrm{d}s\Bigg|,\quad t\in[a,\tau],\ m\ge1.\end{aligned}$$

By assumption, $f\in\mathrm{Lip}_K(\mathcal{O}_\varrho(\Omega_{a,\tau-}))$ and, therefore, in view of (25),

$$r_m(t;\tau,\xi,\lambda)\le K\left(\left(1-\frac{t-a}{\tau-a}\right)\int_a^t r_{m-1}(s;\tau,\xi,\lambda)\,\mathrm{d}s+\frac{t-a}{\tau-a}\int_t^\tau r_{m-1}(s;\tau,\xi,\lambda)\,\mathrm{d}s\right) \tag{29}$$

for $t\in[a,\tau]$, $m\ge2$. Considering (29) and taking (9), (10), (26), and Lemma 2.4 into account, we find, in particular, that

$$\begin{aligned}r_2(t;\tau,\xi,\lambda)&\le\frac{1}{2}K\left[\left(1-\frac{t-a}{\tau-a}\right)\int_a^t\alpha_1(s;a,\tau)\,\mathrm{d}s+\frac{t-a}{\tau-a}\int_t^\tau\alpha_1(s;a,\tau)\,\mathrm{d}s\right]\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f)\\&\le\frac{1}{2}K\alpha_2(t;a,\tau)\,\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f)\\&\le\frac{5}{9}K_*\alpha_1(t;a,\tau)\,\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f)\end{aligned}$$

for all $t\in[a,\tau]$. Similarly, arguing by induction and using (25) and Lemma 2.4, one shows that

$$\begin{aligned}r_m(t;\tau,\xi,\lambda)&\le\frac{1}{2}K^m\alpha_m(t;a,\tau)\,\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f)\\&\le\frac{5}{9}K_*^m\alpha_1(t;a,\tau)\,\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f)\end{aligned} \tag{30}$$

for any $t \in [a, \tau]$ and $m \geq 1$ and, therefore, in view of (28),

$$|x_{m+j}(t;\tau,\xi,\lambda) - x_m(t;\tau,\xi,\lambda)| \leq \sum_{i=1}^{j} r_{m+i}(t;\tau,\xi,\lambda)$$
$$\leq \frac{5}{9}\alpha_1(t;a,\tau)\sum_{i=1}^{j} K_*^{m+i-1}\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f)$$
$$= \frac{5}{9}\alpha_1(t;a,\tau)K_*^m\sum_{i=0}^{j-1} K_*^i\delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f), \qquad t \in [a,\tau],\ m = 0,1,\ldots \tag{31}$$

Assumption (22) means that $r(K_*) < 1$ and, hence,

$$\lim_{m\to\infty} K_*^m = 0_n$$

and

$$\sum_{i=0}^{j-1} K_*^i \leq (1_n - K_*)^{-1}.$$

Therefore, estimate (31) implies that $x_m(t;\tau,\xi,\lambda) \rightrightarrows x_\infty(t;\tau,\xi,\lambda)$ uniformly in $(t,\tau,\lambda) \in [a,\tau]\times[a,b]\times\Omega_a\times\Omega_{\tau-}$. It then follows from Proposition 3.1 that $x_\infty(\cdot;\tau,\xi,\lambda)$ is a solution of problem (17)–(19).

Finally, let us show that this solution is unique in the class of functions with graphs lying in $[a,b]\times\mathcal{O}_\varrho(\Omega_{a,\tau-})$. Assume the existence of another solution $\tilde{x}$ of (17)–(19) such that

$$\{\tilde{x}(t) : t \in [a,\tau]\} \subset \mathcal{O}_\varrho(\Omega_{a,\tau-}). \tag{32}$$

Then $\tilde{x}$ and $x := x_\infty(\cdot;\tau,\xi,\lambda)$ both satisfy Eq. (20). Put $w := |x - \tilde{x}|$. Using (32) and the assumption that $f \in \mathrm{Lip}_K(\mathcal{O}_\varrho(\Omega_{a,\tau-}))$, we obtain

$$w(t) \leq K\left(\left(1 - \frac{t-a}{\tau-a}\right)\int_a^t w(s)\,\mathrm{d}s + \frac{t-a}{\tau-a}\int_t^\tau w(s)\,\mathrm{d}s\right), \qquad t \in [a,\tau], \tag{33}$$

whence, similarly to (30), it follows that

$$w(t) \leq \frac{1}{2}K^m\alpha_m(t;a,\tau)\max_{t\in[a,\tau]} w(t)$$
$$\leq \frac{5}{9}K_*^m\alpha_1(t;a,\tau)\max_{t\in[a,\tau]} w(t), \qquad t \in [a,\tau], \tag{34}$$

for an arbitrary $m \geq 1$. In view of (22) and (24), we have $\lim_{m\to\infty} K_*^m = 0$ and, therefore, (34) implies immediately that $w = 0$, i. e., $\tilde{x}$ coincides with x. □

3.2. Iterations for the after-jump evolution

Modifying equalities (15), (16) and Proposition 3.1 appropriately for the two-point boundary condition at the end points of the interval $[\tau, b]$, we can construct parameterized iterations that should help us to describe the solution *after* the jump (i. e., at the moments of time succeeding τ). More precisely, arguing similarly as above, we arrive at the definition

$$y_0(t;\tau,\lambda,\eta) := \left(1 - \frac{t-\tau}{b-\tau}\right)(\lambda + \gamma(\lambda)) + \frac{t-\tau}{b-\tau}\eta, \tag{35}$$

$$y_m(t;\tau,\lambda,\eta) := y_0(t;\tau,\lambda,\eta) + \int_\tau^t f(s, y_{m-1}(s;\tau,\lambda,\eta))\,\mathrm{d}s - \frac{t-\tau}{b-\tau}\int_\tau^b f(s, y_{m-1}(s;\tau,\lambda,\eta))\,\mathrm{d}s \tag{36}$$

for all $t \in [\tau, b]$ and $m = 0, 1, \ldots$, and easily establish the following proposition.

Proposition 3.3. *Let $\tau \in (a, b)$, $\lambda \in \Omega_{\tau-}$, and $\eta \in \Omega_b$ be fixed. Then*

$$y_m(\tau;\tau,\lambda,\eta) = \lambda + \gamma(\lambda)$$

and

$$y_m(b;\tau,\lambda,\eta) = \eta$$

for any $m \geq 0$. Furthermore, if $\lim_{m\to\infty} y_m(\cdot;\tau,\lambda,\eta) =: y(\cdot)$ exists uniformly on $[\tau, b]$, then $y(\cdot)$ is a solution of the problem

$$y'(t) = f(t, y(t)) + \frac{1}{b-\tau}\left(\eta - \lambda - \gamma(\lambda) - \int_\tau^b f(s, y(s))\,\mathrm{d}s\right), \qquad t \in [\tau, b], \tag{37}$$

$$y(\tau) = \lambda + \gamma(\lambda), \tag{38}$$

$$y(b) = \eta. \tag{39}$$

We see that the function sequence $\{y_m(\cdot; \tau, \lambda, \eta) : m \geq 0\}$ determined by (35) and (36) is a direct analogue of that given by (15), (16). Furthermore, Proposition 3.3 and an argument completely similar to the proof of Theorem 3.2 allow us to obtain the following statement.

Theorem 3.4. *Let there exist a non-negative vector* ϱ *with the property*

$$\varrho \geq \frac{b-a}{4} \delta_{\mathcal{O}_\varrho(\Omega_{\tau+,b})}(f) \tag{40}$$

such that $f \in \mathrm{Lip}_K(\mathcal{O}_\varrho(\Omega_{\tau+,b}))$ *with a certain matrix K satisfying condition (22). Then, for any* $\tau \in (a, b)$, $\lambda \in \Omega_{\tau-}$, *and* $\eta \in \Omega_b$:

1. *Functions (36) are continuously differentiable on* $[\tau, b]$ *for any* $m \geq 0$;
2. $\{y_m(t; \tau, \lambda, \eta) : t \in [\tau, b],\ m \geq 0\} \subset \mathcal{O}_\varrho(\Omega_{\tau+,b})$;
3. $\{y_m(\cdot; \tau, \lambda, \eta) : m \geq 0\}$ *converges to a limit function* $y_\infty(\cdot; \tau, \lambda, \eta)$ *uniformly on* $[\tau, b]$;
4. $y_\infty(\cdot; \tau, \lambda, \eta)$ *is a solution of the boundary value problem (37)–(39) and this problem has no other solutions with values in* $\mathcal{O}_\varrho(\Omega_{\tau+,b})$;
5. *The estimate*

$$|y_\infty(t; \tau, \lambda, \eta) - y_m(t; \tau, \lambda, \eta)| \leq \frac{5}{9}\alpha_1(t; \tau, b) K_*^m (1_n - K_*)^{-1} \delta_{\mathcal{O}_\varrho(\Omega_{\tau+,b})}(f) \tag{41}$$

holds for all $t \in [\tau, b]$ *and* $m \geq 1$, *where* K_* *is given by (24).*

In this way, under the conditions of the above theorems, we can construct the functions $u_m(\cdot; \tau, \xi, \lambda, \eta)$, $m \geq 0$,

$$u_m(t; \tau, \xi, \lambda, \eta) := \begin{cases} x_m(t; \tau, \xi, \lambda) & \text{if } t \leq \tau \\ y_m(t; \tau, \lambda, \eta) & \text{if } t > \tau \end{cases}$$

and claim that the formula

$$u_\infty(t; \tau, \xi, \lambda, \eta) := \begin{cases} x_\infty(t; \tau, \xi, \lambda) & \text{if } t \leq \tau \\ y_\infty(t; \tau, \lambda, \eta) & \text{if } t > \tau \end{cases} \tag{42}$$

introduces a well-defined function $u_\infty : [a, b] \times (a, b) \times \Omega_a \times \Omega_{\tau-} \times \Omega_{\tau+} \to \mathbb{R}^n$ (which, in addition, is known to have values in the set $\mathcal{O}_\varrho(\Omega_{a,\tau-}) \cup \mathcal{O}_\varrho(\Omega_{\tau+,b})$). Function (42), as it turns out, can be used to describe the solutions of the original problem.

4. Determining equations

In order to make clear the relation of function (42) to problem (1)–(3), it is convenient to bring the statements contained in Theorems 3.2 and 3.4 to an alternative form. For this purpose, given arbitrary $\tau \in (a, b)$ and $(\xi, \lambda, \eta) \in \Omega_a \times \Omega_{\tau-} \times \Omega_b$, we put

$$\Xi(\tau, \xi, \lambda) := \lambda - \xi - \int_a^\tau f(s, x_\infty(s; \tau, \xi, \lambda))\, ds, \tag{43}$$

$$\mathrm{H}(\tau, \lambda, \eta) := \eta - \lambda - \gamma(\lambda) - \int_\tau^b f(s, y_\infty(s; \tau, \lambda, \eta))\, ds. \tag{44}$$

The assumptions of the theorems mentioned ensure that formulas (43) and (44) make sense for all the indicated values of variables and thus indeed define mappings from $\Omega_a \times \Omega_{\tau-} \times \Omega_b$ to $\mathbb{R}^n$.

Theorem 4.1. *The following assertions are true.*

1. *Under assumptions of Theorem 3.2, the function* $x_\infty(\cdot; \tau, \xi, \lambda)$ *is a solution of the equation*

$$x'(t) = f(t, x(t)) + \frac{1}{\tau - a}\Xi(\tau, \xi, \lambda), \qquad t \in [a, \tau], \tag{45}$$

satisfying the two-point boundary conditions (18), (19). Problem (45), (18), (19) has no other solutions with values in $\mathcal{O}_\varrho(\Omega_{a,\tau-})$.

2. *Under assumptions of Theorem 3.4, the function* $y_\infty(\cdot; \tau, \lambda, \eta)$ *is a solution of the equation*

$$y'(t) = f(t, y(t)) + \frac{1}{b - \tau}\mathrm{H}(\tau, \lambda, \eta), \qquad t \in [a, \tau], \tag{46}$$

satisfying the two-point boundary conditions (38), (39). Problem (46), (38), (39) has no other solutions with values in $\mathcal{O}_\varrho(\Omega_{\tau+,b})$.

Proof. In view of (43), Eq. (45) for $x := x_\infty(\cdot\,; \tau, \xi, \lambda)$ is a direct consequence of (17). In particular, x satisfies the Cauchy problem (45), (18). If we suppose that another solution $\tilde{x}$ of (45), (18) has values in the set $\mathcal{O}_\varrho(\Omega_{a,\tau-})$, then the assumption $f \in \mathrm{Lip}_K(\mathcal{O}_\varrho(\Omega_{a,\tau-}))$ leads one to the standard estimate

$$|\tilde{x}(t) - x(t)| \le K^m \frac{(t-a)^m}{m!} \max_{s\in[a,\tau]} |\tilde{x}(s) - x(s)| \tag{47}$$

for all $t \in [a, \tau]$ and $m \ge 1$, which implies immediately that $\tilde{x}$ should coincide with x. The case of problem (46), (38), (39) is considered by analogy. □

Eqs. (45) and (46) are forced versions of the original equation (1). The next statement allows one to characterize expressions (43) and (44) as optimal, in a sense, values of the forcing term.

Theorem 4.2. *Let $\tau \in (a, b)$, $\lambda \in \Omega_{\tau-}$, $\eta \in \Omega_b$, and $\mu \in \mathbb{R}^n$ be fixed.*

1. *Let there exist ϱ and K such that assumptions of Theorem 3.2 hold. Then a solution $x(\cdot)$ of the equation*

$$x'(t) = f(t, x(t)) + \frac{1}{\tau - a}\mu \qquad t \in [a, \tau], \tag{48}$$

has values in $\mathcal{O}_\varrho(\Omega_{a,\tau-})$ and satisfies the two-point boundary conditions (18), (19) if, and only if

$$\mu = \Xi(\tau, \xi, \lambda). \tag{49}$$

2. *Let there exist ϱ and K such that assumptions of Theorem 3.4 hold. Then a solution $y(\cdot)$ of the equation*

$$y'(t) = f(t, y(t)) + \frac{1}{b - \tau}\mu \qquad t \in [\tau, b], \tag{50}$$

has values in $\mathcal{O}_\varrho(\Omega_{\tau+,b})$ and satisfies the two-point boundary conditions (38), (39) if, and only if

$$\mu = \mathrm{H}(\tau, \lambda, \eta). \tag{51}$$

Proof. We shall prove, e. g., the first assertion concerning Eq. (48) (the case of Eq. (50) is considered by analogy).

Let $\tau \in (a, b)$, $\lambda \in \Omega_{\tau-}$, $\eta \in \Omega_b$, and $\mu \in \mathbb{R}^n$. Under conditions of Theorem 3.2, the function $x_\infty(\cdot\,; \tau, \xi, \lambda)$ and, hence, the corresponding expression (43) are well defined. If μ in (48) is given by formula (49), then (48) is nothing but (45). By Theorem 4.1, the function $x_\infty(\cdot\,; \tau, \xi, \lambda)$ is a solution of problem (48), (18), (19) and its values do not escape from the set $\mathcal{O}_\varrho(\Omega_{a,\tau-})$. Moreover, the same theorem ensures that, in this case, (48) does not have any other solutions with properties (18), (19) and graphs lying in $[a, b] \times \mathcal{O}_\varrho(\Omega_{a,\tau-})$.

Conversely, let us fix a certain μ and assume that problem (48), (18), (19) has a solution x with values in

$$\mathcal{O}_\varrho(\Omega_{a,\tau-}).$$

We thus have

$$x(t) = \xi + \int_a^t f(s, x(s))\,\mathrm{d}s - \frac{t-a}{\tau - a}\mu, \qquad t \in [a, \tau], \tag{52}$$

whence it follows that μ satisfies the relation

$$\mu = x(\tau) - \xi - \int_a^\tau f(s, x(s))\,\mathrm{d}s. \tag{53}$$

Substituting (53) into (52) and recalling that, by assumption, $x(\tau) = \lambda$, we get

$$x'(t) = f(s, x(t)) - \frac{1}{\tau - a}\left(\lambda - \xi - \int_a^\tau f(s, x(s))\,\mathrm{d}s\right), \qquad t \in [a, \tau]. \tag{54}$$

It follows from (54) that x is a solution of (17)– (19). Therefore, by Theorem 3.2,

$$x = x_\infty(\cdot\,; \tau, \xi, \lambda)$$

since problem (17)–(19) does not have any solutions different from $x_\infty(\cdot\,; \tau, \xi, \lambda)$. Reusing (53), we arrive immediately at (49). □

Although the explicit form of the mappings Ξ and H is unknown, it is noteworthy that, in contrast to (17) and (37), Eqs. (45) and (46) are ordinary differential equations that differ from the original equation (1) by a constant forcing term. This simple observation allows us to prove that the functions $x_\infty(\cdot\,; \tau, \xi, \lambda)$ and $y_\infty(\cdot\,; \tau, \xi, \lambda)$ are related to the original impulsive boundary value problem (1)–(3) in the following way.

Theorem 4.3. *Let there exist two pairs (ϱ_0, K_0) and (ϱ_1, K_1) satisfying, respectively, conditions of Theorems 3.2 and 3.4. Then the following assertions hold.*

1. *If the equalities*

$$\begin{aligned} \Xi(\tau,\xi,\lambda)&=0,\\ \mathrm{H}(\tau,\lambda,\eta)&=0,\\ g(\tau,\lambda)&=0,\\ A\xi+C\eta&=d \end{aligned} \tag{55}$$

hold for certain values $(\tau,\xi,\lambda,\eta)\in(a,b)\times\Omega_a\times\Omega_{\tau-}\times\Omega_b$ *and, in addition,*

$$g(t,y_\infty(t;\tau,\lambda,\eta))\neq 0\quad \text{for any } t\in(\tau,b], \tag{56}$$

then the function $u_\infty(\cdot;\tau,\xi,\lambda,\eta)$ *given by (42) is a solution of problem (1)–(3) with exactly one jump at the instant of time* τ.

2. *If* $u(\cdot)$ *is a solution of problem (1)–(3) with a jump at a time instant* τ *such that*

$$\begin{aligned} &\{u(t):t\in[a,\tau]\}\subset\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}),\quad \{u(t):t\in[\tau,b]\}\subset\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b}),\\ &u(a)\in\Omega_a,u(\tau)\in\Omega_{\tau-},\quad u(b)\in\Omega_b, \end{aligned} \tag{57}$$

then $(\tau,u(a),u(\tau),u(b))$ *necessarily satisfy system (55). If, moreover, this solution has no other jumps, then (56) also holds for the values indicated.*

Proof. 1. Let $\tau\in(a,b)$, $\xi\in\Omega_a$, $\lambda\in\Omega_{\tau-}$, and $\eta\in\Omega_b$ and let $u:=u_\infty(\cdot;\tau,\xi,\lambda,\eta)$ be defined according to (42). By Theorem 4.1, the restrictions $x:=u|_{[a,\tau]}$ and $y:=u|_{(\tau,b]}$ of u to the corresponding subintervals have continuous derivatives and satisfy Eqs. (45) and (46) respectively. Assume that (55) holds for the parameters (τ,ξ,λ,η). The first two equations of (55) then imply that

$$u'(t)=f(t,u(t)),\qquad t\in[a,b]\setminus\{\tau\}.$$

Since x and y satisfy, respectively, conditions (18), (19) and (38), (39), we obtain

$$u(a)=\xi,\qquad u(\tau-)=u(\tau)=\lambda,\qquad u(\tau+)=\lambda+\gamma(\lambda),\qquad u(b)=\eta, \tag{58}$$

whence it follows, in particular, that $u(\tau+)-u(\tau-)=\gamma(u(\tau-))$. Due to the third equation in (55), we also have $g(\tau,u(\tau-))=0$. This means that u satisfies the jump condition (2) for $t=\tau$ and, therefore, the point $(\tau,u(\tau))$ belongs to the barrier set G given by (4). Considering assumption (56), we conclude that the solution does not meet G at any other point. Finally, the last equation in (55) guarantees that u fulfils the boundary conditions (3). We have thus proved that u is a solution of (1)–(3) with $p=1$ in Definition 1.1.

2. Let u be a solution of (1)–(3) with exactly one jump and let (57) hold. Then there exists a unique point $\tau\in(a,b)$ such that the restrictions $x:=u|_{[a,\tau]}$ and $y:=u|_{(\tau,b]}$ of u to the respective subintervals have continuous derivatives and u has a unique jump at τ so that

$$u(\tau+)-u(\tau)=\gamma(u(\tau)).$$

Thus, the functions x and y have the properties

$$\begin{aligned} &x(a)=u(a), && x(\tau)=u(\tau),\\ &y(\tau)=u(\tau)+\gamma(u(\tau)), && y(b)=u(b) \end{aligned}$$

and satisfy respectively Eqs. (48) and (50) with the zero value of μ. In other words, x and y are solutions of the respective problems (48), (18), (19) and (50), (38), (39) with $\mu=0$ and

$$\xi:=u(a),\quad \lambda:=u(\tau),\quad \eta:=u(b). \tag{59}$$

Theorem 4.2 then implies that values (59) of (ξ,λ,η) necessarily satisfy the first two equations in (55). Recalling now Theorem 4.1, we conclude that $x=x_\infty(\cdot;\tau,\xi,\lambda)$ and $y=y_\infty(\cdot;\tau,\lambda,\eta)$ with this choice of parameters, whence the rest of the assertions easily follows. □

Remark 4.4. System (55) consists of $3n+1$ scalar equations for $3n+1$ scalar unknown parameters τ, $\xi_1,\ldots,\xi_n$, $\lambda_1,\ldots,\lambda_n$, $\eta_1,\ldots,\eta_n$, i. e., the number of equations coincides with the number of unknowns involved.

In other words, under conditions of Theorem 4.3, system (55), (56) allows one to determine all possible solutions u of problem (1)–(3) having exactly one jump and possessing properties (57).

Remark 4.5. The simplest way to choose the parameter sets, which also seems to be sufficient for most applications, is to take a compact convex set $\Omega_a\subset\mathbb{R}^n$ and put

$$\Omega_b:=\{x+\gamma(x):x\in\Omega_a\},\quad \Omega_{\tau-}:=\Omega_a,\quad \Omega_{\tau+}:=\Omega_b. \tag{60}$$

Then, according to (7) and (14), the sets involved in conditions (21) and (40) have the form $\Omega_{a,\tau-}=\Omega_a$ and $\Omega_{\tau+,b}=\Omega_b$ respectively.

Remark 4.6. The argument based on Theorem 4.3 allows one to deal with multiple solutions of the problem. Let (60) and the assumptions of Theorems 3.2 and 3.4 be satisfied. Suppose that system (55) has two distinct solutions $(\tau_1$, ξ_1,

Table 1
Meaning of variables in (63).

The expression	The value it approximates
$\hat{\xi} \in \Omega_a$	$u(a)$
$\hat{\tau} \in (a, b)$	τ
$\hat{\lambda} \in \Omega_{\tau-}$	$u(\tau)$
$\hat{\lambda} + \gamma(\hat{\lambda}) \in \Omega_{\tau+}$	$u(\tau+)$
$\hat{\eta} \in \Omega_b$	$u(b)$

$\lambda_1, \eta_1)$ and $(\tau_2, \xi_2, \lambda_2, \eta_2)$, in the set $(a, b) \times \Omega_a \times \Omega_a \times \Omega_b$. Then we get from Theorems 3.2 and 3.4 the functions $x_\infty(\cdot; \tau_1, \xi_1, \lambda_1), y_\infty(\cdot; \tau_1, \lambda_1, \eta_1), x_\infty(\cdot; \tau_2, \xi_2, \lambda_2)$, and $y_\infty(\cdot; \tau_2, \lambda_2, \eta_2)$.

Finally, assume that

$$g(t, y_\infty(t; \tau_i, \lambda_i, \eta_i)) \neq 0, \qquad t \in (\tau_i, b],$$

for $i = 1, 2$. Then problem (1)–(3) has two distinct solutions u_1 and u_2 which can be represented as

$$u_i(t) = \begin{cases} x_\infty(t; \tau_i, \xi_i, \lambda_i) & \text{if } t \in [a, \tau_i], \\ y_\infty(t; \tau_i, \lambda_i, \eta_i) & \text{if } t \in (\tau_i, b] \end{cases}$$

for $i = 1, 2$.

5. Approximation of solutions

The solvability of the determining system (55) can be established similarly to [39] by studying its approximate version

$$\begin{aligned} &\int_a^\tau f(s, x_m(s; \tau, \xi, \lambda))\,\mathrm{d}s = \lambda - \xi, \\ &\int_\tau^b f(s, y_m(s; \tau, \lambda, \eta))\,\mathrm{d}s = \eta - \lambda - \gamma(\lambda), \\ &\qquad g(\tau, \lambda) = 0, \\ &\qquad Az + C\eta = d \end{aligned} \tag{61}$$

with the additional condition

$$g(t, y_m(t; \tau, \lambda, \eta)) \neq 0, \qquad t \in (\tau, b]. \tag{62}$$

Clearly, (61), (62) is obtained from (55), (56) by replacing the limit function by one of the iterations. It is important that the value of $m \geq 0$ here is fixed and all the terms involved in (61) and (62) can be constructed explicitly.

Let the quartet $(\hat{\tau}, \hat{\xi}, \hat{\lambda}, \hat{\eta}) \in (a, b) \times \Omega_a \times \Omega_{\hat{\tau}} \times \Omega_b$ be a root of system (61) for a fixed $m \geq 0$. Assume that $(\hat{\tau}, \hat{\xi}, \hat{\lambda}, \hat{\eta})$ also satisfy (62) and consider the function

$$\hat{u}(t) := \begin{cases} x_m(t; \hat{\tau}, \hat{\xi}, \hat{\lambda}) & \text{if } t \in [a, \hat{\tau}], \\ y_m(t; \hat{\tau}, \hat{\lambda}, \hat{\eta}) & \text{if } t \in (\hat{\tau}, b]. \end{cases} \tag{63}$$

Due to (62), the function $\hat{u}$ undergoes a single jump of value $\gamma(\hat{\lambda})$ at the instant of time $\hat{\tau}$. Recalling Theorem 3.2, we obtain that inequality (23) yields

$$|x_\infty(t; \hat{\tau}, \hat{\xi}, \hat{\lambda}) - x_m(t; \hat{\tau}, \hat{\xi}, \hat{\lambda})| \leq \frac{5}{9}\alpha_1(t; a, \hat{\tau}) K_*^m (1_n - K_*)^{-1} \delta_{\mathcal{O}_\varrho(\Omega_{a,\tau-})}(f), \qquad t \in [a, \hat{\tau}],\ m \geq 1, \tag{64}$$

with K and ϱ from Theorem 3.2. In a similar manner, inequality (41) implies that

$$|y_\infty(t; \hat{\tau}, \hat{\lambda}, \hat{\eta}) - y_m(t; \hat{\tau}, \hat{\lambda}, \hat{\eta}| \leq \frac{5}{9}\alpha_1(t; \hat{\tau}, b) K_*^m (1_n - K_*)^{-1} \delta_{\mathcal{O}_\varrho(\Omega_{\tau+,b})}(f), \qquad t \in [\hat{\tau}, b],\ m \geq 1, \tag{65}$$

with K and ϱ from Theorem 3.4.

Estimates (64) and (65) allow one to regard function (63) as the mth approximation to a solution of problem (1)–(3). The meaning of the variables appearing in (63) in relation to the exact solution u of (1)–(3) is explained by Table 1.

The solvability analysis based on properties of equations (61) can be carried out by analogy to [41,42] based on topological degree methods. This topic is not treated here.

The most difficult part of the approach presented here is the construction of the functions $x_m(\cdot; \tau, \xi, \lambda)$ and $y_m(\cdot; \tau, \lambda, \eta)$ in (16) and (36). If the explicit integration in (16) and (36) is impossible or difficult, one can use suitable modified versions of (16) and (36) which, at the expense of a certain loss in accuracy, lead one to iterations better suited for practical computations. Two modifications of this kind are particularly natural, namely, a scheme with polynomial interpolation [43] and that which may be called as “frozen” parameters scheme, where the computation is facilitated by substituting the roots of the $(m-1)$th approximate determining system into the formula for the mth iteration *before* the mth determining system is constructed (see, e. g. [34]). In the latter case, where the "frozen" parameters modification is implemented, one can suggest the following algorithm for the approximate solution of problem (1)–(3).

1. Choose a compact convex set $\Omega_a \subset \mathbb{R}^n$ and put $\Omega_b = \{x + \gamma(x) : x \in \Omega_a\}$ (see (60) in Remark 4.5). Verify the assumptions of Theorems 3.2 and 3.4 and, in case of success, continue to the next step.
2. Compute $x_1(\cdot; \tau, \xi, \lambda)$ and $y_1(\cdot; \tau, \lambda, \eta)$ from (16) and (36) keeping $(\tau, \xi, \lambda, \eta) \in (a, b) \times \Omega_a \times \Omega_a \times \Omega_b$ as parameters.
3. Put $m := 1$ in system (61) and find its solution $(\hat{\tau}, \hat{\xi}, \hat{\lambda}, \hat{\eta}) \in (a, b) \times \Omega_a \times \Omega_a \times \Omega_b$.
4. Put $m := 2$, $X_1 := x_1(\cdot; \hat{\tau}, \hat{\xi}, \hat{\lambda})$ and $Y_1 := y_1(\cdot; \hat{\tau}, \hat{\lambda}, \hat{\eta})$ and, for arbitrary parameters $(\tau, \xi, \lambda, \eta) \in (a, b) \times \Omega_a \times \Omega_a \times \Omega_b$, derive the second "frozen" iterations $\hat{x}_2(\cdot; \tau, \xi, \lambda)$ and $\hat{y}_2(\cdot; \tau, \lambda, \eta)$ by inserting the functions X_1 and Y_1 into (16) and (36) as follows:

$$\hat{x}_2(t; \tau, \xi, \lambda) := \xi + \int_a^t f(s, X_1(s))\,ds - \frac{t-a}{\tau-a}\int_a^\tau f(s, X_1(s))\,ds + \frac{t-a}{\tau-a}(\lambda - \xi) \qquad \text{for } t \in [a, \tau] \text{ and} \tag{66}$$

$$\hat{y}_2(t; \tau, \lambda, \eta) := \lambda + \gamma(\lambda) + \int_\tau^t f(s, Y_1(s))\,ds - \frac{t-\tau}{b-\tau}\int_\tau^b f(s, Y_1(s))\,ds + \frac{t-\tau}{b-\tau}(\eta - \lambda - \gamma(\lambda)) \tag{67}$$

   for $t \in [\tau, b]$.
5. For $m = 2$, modify system (61) by substituting there the second "frozen" iterations $\hat{x}_2(\cdot; \tau, \xi, \lambda)$ and $\hat{y}_2(\cdot; \tau, \lambda, \eta)$. Solve the resulting modified approximate determining system

$$\begin{aligned} &\int_a^\tau f(s, \hat{x}_2(s; \tau, \xi, \lambda))\,ds = \lambda - \xi, \\ &\int_\tau^b f(s, \hat{y}_2(s; \tau, \lambda, \eta))\,ds = \eta - \lambda - \gamma(\lambda), \\ &\qquad g(\tau, \lambda) = 0, \\ &\qquad Az + C\eta = d \end{aligned} \tag{68}$$

   in the set $(a, b) \times \Omega_a \times \Omega_a \times \Omega_b$ and denote its solution again by $(\hat{\tau}, \hat{z}, \hat{\lambda}, \hat{\eta})$.
6. Similarly to (66) and (67), construct the third "frozen" iterations $\hat{x}_3(\cdot; \tau, \xi, \lambda)$ and $\hat{y}_3(\cdot; \tau, \lambda, \eta)$ by inserting the functions $X_2 := x_2(\cdot; \hat{\tau}, \hat{\xi}, \hat{\lambda})$ and $Y_2 := y_2(\cdot; \hat{\tau}, \hat{\lambda}, \hat{\eta})$ into (16) and (36) for $m = 3$. The values $(\tau, \xi, \lambda, \eta) \in (a, b) \times \Omega_a \times \Omega_a \times \Omega_b$ are kept as parameters.
7. For $m = 3$, modify system (61) by inserting there the third "frozen" iterations $\hat{x}_3(\cdot; \tau, \xi, \lambda)$ and $\hat{y}_3(\cdot; \tau, \lambda, \eta)$. Solve the resulting modified system in the set $(a, b) \times \Omega_a \times \Omega_a \times \Omega_b$ and denote its solution again by $(\hat{\tau}, \hat{\xi}, \hat{\lambda}, \hat{\eta})$.
8. In a similar manner, derive several further "frozen" iterations.
9. If, for some $m \geq 0$, the mth and $(m-1)$th "frozen" iterations are close enough to one another, verify the condition

$$g(t, Y_m(t)) \neq 0, \qquad t \in (\hat{\tau}, b]. \tag{69}$$

   If (69) is fulfilled, then the function

$$\hat{u}(t) := \begin{cases} X_m(t) & \text{if } t \in [a, \hat{\tau}], \\ Y_m(t) & \text{if } t \in (\hat{\tau}, b], \end{cases}$$

   is regarded as the mth approximation of a solution u of problem (1)–(3) with $u(a) \in \Omega_a$ and exactly one jump. If (69) is not satisfied, then a different set Ω_a should be chosen.

Note that the computation can be started directly at $m = 0$, in which case no iteration is carried out yet and one works with the initial functions (15), (35) only. Being a piecewise linear function, the zeroth approximation is very rough but, nevertheless, it is usually helpful as a preliminary "shot" providing us with a certain basic information. In particular, the roots of the zeroth approximate determining equation, which has the simplest possible form because linear functions of ξ, λ and η are substituted into the non-linearity, can provide a hint helping one to choose the sets Ω_a, $\Omega_{\tau-}$, $\Omega_{\tau+}$, and Ω_b in a suitable way and avoid unnecessary computations on sets that might possibly be excessively large.

6. Example

Let us apply the numerical-analytic approach described above to the system

$$\begin{aligned} u_1'(t) &= (u_2(t))^2 - \frac{t}{5}u_1(t) + \frac{t^3}{100} - \frac{t^2}{25}, \\ u_2'(t) &= \frac{t^2}{10}u_2(t) + \frac{t}{8}u_1(t) - \frac{21t^3}{800} + \frac{t}{16} + \frac{1}{5}, \quad \text{a.e. } t \in \left[0, \frac{1}{2}\right], \end{aligned} \tag{70}$$

with state-dependent jumps governed by the rule

$$u_1(t+)-u_1(t-)=\frac{1}{2},\quad u_2(t+)-u_2(t-)=-\frac{1}{10}\quad \text{for } t \text{ such that } \left(u_1(t)+\frac{1}{2}\right)^2+u_2(t)=\frac{1}{25}. \tag{71}$$

Let us consider the impulsive system (70), (71) under the two-point boundary condition

$$\begin{pmatrix}\frac{1}{4} & -\frac{1}{2}\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1(0)\\ u_2(0)\end{pmatrix}+\begin{pmatrix}\frac{1}{2} & 0\\ \frac{1}{4} & 0\end{pmatrix}\begin{pmatrix}u_1\left(\frac{1}{2}\right)\\ u_2\left(\frac{1}{2}\right)\end{pmatrix}=\begin{pmatrix}-0.1212\\ 0.0019\end{pmatrix}. \tag{72}$$

We are interested in solutions of problem (70)–(72) according to Definition 1.1 with $p=1$. Therefore, our solution is a left-continuous vector function $u:\left[0,\frac{1}{2}\right]\to\mathbb{R}^2$, $u=\mathrm{col}(u_1,u_2)$, which intersects the barrier

$$G:=\left[0,\tfrac{1}{2}\right]\times\left\{(x_1,x_2)\in\mathbb{R}^2:\left(x_1+\frac{1}{2}\right)^2+x_2-\frac{1}{25}=0\right\} \tag{73}$$

exactly once. Accordingly, there exists a unique $\tau\in\left(0,\frac{1}{2}\right)$ such that

$$\left(u_1(\tau)+\frac{1}{2}\right)^2+u_2(\tau)=\frac{1}{25}. \tag{74}$$

Furthermore, the restrictions $u|_{[0,\tau]}$ and $u|_{(\tau,b]}$ have continuous derivatives, u satisfies (70) for $t\in\left[0,\frac{1}{2}\right]\setminus\{\tau\}$ and has a jump at τ. The size of the jump is given by the constant vector $\gamma=\mathrm{col}\,(0.5,-0.1)$. Finally, u satisfies (72).

We describe in detail the individual steps of our method. Let $a=0$, $b=1/2$, and $f=\mathrm{col}(f_1,f_2)$, where

$$f_1(t,x_1,x_2)=x_2^2-\frac{t}{5}x_1+\frac{t^3}{100}-\frac{t^2}{25},\quad f_2(t,x_1,x_2)=\frac{t^2}{10}x_2+\frac{t}{8}x_1-\frac{21t^3}{800}+\frac{t}{16}+\frac{1}{5} \tag{75}$$

for all $t\in\left[0,\frac{1}{2}\right]$ and $(x_1,x_2)\in\mathbb{R}^2$. Clearly, (70)–(72) is a particular case of (1)–(3).

6.1. Application of Theorem 3.2

Assume that $\tau\in(0,1/2)$ is a parameter and put

$$\Omega_0=\Omega_{\tau-}=\left\{(x_1,x_2):-8.44\le x_1\le 0.15,\ -4.0\le x_2\le 0.15\right\}. \tag{76}$$

Then the corresponding set $\Omega_{0,\tau-}$ coincides with (76) (see Remark 4.5). Let us put, e. g.,

$$\varrho_0=\mathrm{col}\,(2.46,0.2). \tag{77}$$

Then, by (6) and (76), the ϱ_0-neighborhood $\mathcal{O}_{\varrho_0}(\Omega_{0,\tau-})$ of the set $\Omega_{0,\tau-}$ (or, which is the same in this case, Ω_0) has the form

$$\mathcal{O}_{\varrho_0}(\Omega_0)=\left\{(x_1,x_2):-10.9\le x_1\le 2.61,\ -4.2\le x_2\le 0.35\right\}.$$

Direct computations show that f given by (75) belongs to $\mathrm{Lip}_{K_0}(\mathcal{O}_{\varrho_0}(\Omega_0))$ (see (8)) with ϱ_0 given by (77) and

$$K_0:=\begin{pmatrix}\frac{1}{10} & \frac{42}{5}\\ \frac{1}{16} & \frac{1}{40}\end{pmatrix}.$$

Since $r(K_0)\approx 0.788$, it follows that $r(K_0)<10/(3(b-a))=20/3$, i. e., K_0 satisfies (22). Computing the vector $\delta_{\mathcal{O}_{\varrho_0}(\Omega_0)}(f)$ according to (9) and (75), we find

$$\delta_{\mathcal{O}_{\varrho_0}(\Omega_0)}(f)=\max_{(t,x)\in\left[0,\frac{1}{2}\right]\times\mathcal{O}_{\varrho_0}(\Omega_0)}f(t,x)-\min_{(t,x)\in\left[0,\frac{1}{2}\right]\times\mathcal{O}_{\varrho_0}(\Omega_0)}f(t,x)=\begin{pmatrix}18.991\\ 0.958125\end{pmatrix},$$

and, therefore,

$$\varrho_0=\begin{pmatrix}2.46\\ 0.2\end{pmatrix}\ge\begin{pmatrix}2.373875000\\ 0.119765625\end{pmatrix}=\frac{1}{8}\delta_{\mathcal{O}_{\varrho_0}(\Omega_0)}(f). \tag{78}$$

Inequality (78) means that (21) holds. All the conditions of Theorem 3.2 are thus satisfied and, consequently, the sequence of functions (16) is convergent in this example.

6.2. Application of Theorem 3.4

Let us put

$$\Omega_{\frac{1}{2}}=\Omega_{\tau+}=\left\{(y_1,y_2):-7.94\le y_1\le 0.7,\ -4.15\le y_2\le 0.05\right\}. \tag{79}$$

According to Remark 4.5, $\Omega_{\tau+,\frac{1}{2}}$ coincides with set (79). Choose the vector

$$\varrho_1 := \text{col}(2.63, 0.15).$$

Then, according to (6) and (76), the ϱ_1-neighborhood $\mathcal{O}_{\varrho_1}(\Omega_{\frac{1}{2}})$ of $\Omega_{\frac{1}{2}}$ has the form

$$\mathcal{O}_{\varrho_1}(\Omega_{\frac{1}{2}}) = \left\{(y_1, y_2) : -10.57 \le y_1 \le 3.33, -4.3 \le y_2 \le 0.2\right\}.$$

A direct computation shows that $f \in \text{Lip}_{K_1}(\mathcal{O}_{\varrho_1}(\Omega_{\frac{1}{2}}))$ with

$$K_1 = \begin{pmatrix} 1/10 & 43/5 \\ 1/16 & 1/40 \end{pmatrix}.$$

Since $r(K_1) \approx 0.7966$, we have $r(K_1) < 10/(3(b-a)) = 20/3$ and, hence, K_1 satisfies (22). Furthermore, (9) yields

$$\frac{1}{8}\delta_{\mathcal{O}_{\varrho_1}(\Omega_{1/2})}(f) = \frac{1}{8}\begin{pmatrix} 19.88 \\ 0.98125 \end{pmatrix} = \begin{pmatrix} 2.485 \\ 0.12265625 \end{pmatrix} \le \begin{pmatrix} 2.63 \\ 0.15 \end{pmatrix} = \varrho_1$$

which means that ϱ_1 satisfies condition (40). Thus, all the assumptions of Theorem 3.4 hold and the convergence of sequence (36) is guaranteed.

6.3. Starting functions and first iteration

Consider the parameters $(\tau, \xi, \lambda, \eta) \in \left(0, \frac{1}{2}\right) \times \Omega_0 \times \Omega_0 \times \Omega_{\frac{1}{2}}$, where

$$\xi = \text{col}(\xi_1, \xi_2),$$

$$\lambda = \text{col}(\lambda_1, \lambda_2)$$

,

$$\eta = \text{col}(\eta_1, \eta_2).$$

By (15) and (35), the starting functions $x_0 = \text{col}(x_{01}, x_{02})$ and $y_0 = \text{col}(y_{01}, y_{02})$ have the form

$$x_{01}(t; \tau, \xi_1, \lambda_1) = \left(1 - \frac{t}{\tau}\right)\xi_1 + \frac{t}{\tau}\lambda_1, \tag{80}$$

$$x_{02}(t; \tau, \xi_2, \lambda_2) = \left(1 - \frac{t}{\tau}\right)\xi_2 + \frac{t}{\tau}\lambda_2 \tag{81}$$

for $t \in [0, \tau]$ and

$$y_{01}(t; \tau, \lambda_1, \eta_1) = \left(1 - \frac{t-\tau}{\frac{1}{2}-\tau}\right)(\lambda_1 + 0.5) + \frac{t-\tau}{\frac{1}{2}-\tau}\eta_1, \tag{82}$$

$$y_{02}(t; \tau, \lambda_2, \eta_2) = \left(1 - \frac{t-\tau}{\frac{1}{2}-\tau}\right)(\lambda_2 - 0.1) + \frac{t-\tau}{\frac{1}{2}-\tau}\eta_2 \tag{83}$$

for $t \in [\tau, \frac{1}{2}]$. The first iterations $x_1 = \text{col}(x_{11}, x_{12})$ and $y_1 = \text{col}(y_{11}, y_{12})$ can be found by using symbolic computation systems (in our case, Maple 14) from (16) and (36), where $m = 1$, $a = 0$, $b = 1/2$. The explicit expressions may be rather complicated. For example, $x_{11}(\cdot; \tau, \xi_1, \xi_2, \lambda_1, \lambda_2)$ is given by the formula

$$\begin{aligned} x_{11}(t; \tau, \xi_1, \xi_2, \lambda_1, \lambda_2) = \xi_1 &+ \frac{1}{400}t^4 + \frac{1}{3}\left[\left(\frac{-\xi_2}{\tau} + \frac{\lambda_2}{\tau}\right)^2 + \frac{\xi_1}{5\tau} - \frac{\lambda_1}{\tau} - \frac{1}{25}\right]t^3 + \frac{1}{2}\left[2\xi_2\left(\frac{-\xi_2}{\tau} + \frac{\lambda_2}{\tau}\right) - \frac{\xi_1}{5}\right]t^2 \\ &+ \xi_2^2 t - \frac{t}{\tau}\left[\frac{1}{400}\tau^4 + \frac{1}{3}\left(\left(\frac{-\xi_2}{\tau} + \frac{\lambda_2}{\tau}\right)^2 + \frac{\xi_1}{\tau} - \frac{\lambda_1}{5\tau} - \frac{1}{25}\right)\tau^3\right. \\ &\left. + \frac{1}{2}\left(2\xi_2\left(\frac{-\xi_2}{\tau} + \frac{\lambda_2}{\tau}\right) - \frac{\xi_1}{5}\right)\tau^2 + \xi_2^2\tau\right] + \frac{t(\lambda_1 - \xi_1)}{\tau}, \qquad t \in [0, \tau]. \end{aligned}$$

However, once there is a way to construct these functions, the concrete explicit formulas are unimportant at this point because they become proper approximations only after the substitution of suitable values of parameters. Those values are obtained by solving the approximate determining equations. In the case considered, system (61) for $m = 1$ has the form

$$\int_0^\tau f_1(s; x_{11}(s; \tau, \xi_1, \xi_2, \lambda_1, \lambda_2), x_{12}(s; \tau, \xi_1, \xi_2, \lambda_1, \lambda_2))\,ds = \lambda_1 - \xi_1, \tag{84}$$

Table 2
Approximate values of parameters for the first solution of problem (70)–(72).

Variable	$m = 1$	$m = 2$	$m = 3$	$m = 4$
τ	0.377367167	0.377366182	0.377366354	0.377366355
ξ_1	−8.437535639	−8.437471330	−8.437478608	−8.437478618
ξ_2	−3.968767820	−3.968735665	−3.968739304	−3.968739309
λ_1	−2.493949925	−2.493944384	−2.493945315	−2.493945318
λ_2	−3.935836303	−3.935836303	−3.935817921	−3.935817931
η_1	0.007600000	0.007600000	0.007600000	0.007600002
η_2	−4.024145297	−4.024123042	−4.024126787	−4.024126798

$$\int_0^{\tau} f_2(s; x_{11}(s;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2), x_{12}(s;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2))\,\mathrm{d}s = \lambda_2 - \xi_2, \tag{85}$$

$$\int_\tau^{\frac{1}{2}} f_1(s; y_{11}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2), y_{12}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = \eta_1 - \lambda_1 - 0.5, \tag{86}$$

$$\int_\tau^{\frac{1}{2}} f_2(s; y_{11}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2), y_{12}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = \eta_2 - \lambda_2 + 0.1, \tag{87}$$

$$(\lambda_1 + 0.5)^2 + \lambda_2 = 0.04, \tag{88}$$

$$\begin{pmatrix} 0.25 & -0.5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0.5 & 0 \\ 0.25 & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -0.1212 \\ 0.0019 \end{pmatrix}. \tag{89}$$

We see that system (84)–(89) consists of seven algebraic equations with the unknowns τ, ξ_1, ξ_2, λ_1, λ_2, η_1, η_2, the values of which are sought for in the set $(0, \frac{1}{2}) \times \Omega_0 \times \Omega_0 \times \Omega_{\frac{1}{2}}$ (see (76) and (79)). For $\xi_1 \in [-8.44, -1]$, numerical computations give the roots constituting the first column of Table 2. Substituting these roots into $x_{11}(\cdot;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2), x_{12}(\cdot;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2), y_{11}(\cdot;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2)$, and $y_{12}(\cdot;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2)$, we get the respective functions

$$\begin{aligned}
X_{11}(t) &= -8.437535639 + 15.71336319\,t + 0.4974130077\,t^2 - 1.060804186\,t^3 + 0.0025\,t^4, \\
X_{12}(t) &= -3.968767820 + 0.200096725\,t - 0.4960959775\,t^2 + 0.5239635733\,t^3 - 0.004380837\,t^4, \\
Y_{11}(t) &= -8.254109890 + 16.58414073\,t + 0.4271345180\,t^2 - 1.098402835\,t^3 + 0.0025\,t^4, \\
Y_{12}(t) &= -4.072409823 + 0.2001284496\,t - 0.4783214004\,t^2 + 0.5443348006\,t^3 - 0.004179165\,t^4,
\end{aligned}$$

which are used at the next step for the computation of the second iteration.

6.4. Second iteration

Assume again that parameters $(\tau, \xi, \lambda, \eta) \in \left(0, \frac{1}{2}\right) \times \Omega_0 \times \Omega_0 \times \Omega_{\frac{1}{2}}$ are arbitrary. We shall use the "frozen" iterations described in Section 5. Based on the functions $X_1 = \mathrm{col}(X_{11}, X_{12})$ and $Y_1 = \mathrm{col}(Y_{11}, Y_{12})$, we construct the second iterations $\hat{x}_2 = \mathrm{col}(\hat{x}_{21}, \hat{x}_{22})$ and $\hat{y}_2 = \mathrm{col}(\hat{y}_{21}, \hat{y}_{22})$ from (66) and (67), where $a = 0$, $b = 1/2$. Then, according to (68), solve the system

$$\begin{aligned}
&\int_0^{\tau} f_1(s; \hat{x}_{21}(s;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2), \hat{x}_{22}(s;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2))\,\mathrm{d}s = \lambda_1 - \xi_1, \\
&\int_0^{\tau} f_2(s; \hat{x}_{21}(s;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2), \hat{x}_{22}(s;\tau,\xi_1,\xi_2,\lambda_1,\lambda_2))\,\mathrm{d}s = \lambda_2 - \xi_2, \\
&\int_\tau^{\frac{1}{2}} f_1(s; \hat{y}_{21}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2), \hat{y}_{22}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = \eta_1 - \lambda_1 - 0.5, \\
&\int_\tau^{\frac{1}{2}} f_2(s; \hat{y}_{21}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2), \hat{y}_{22}(s;\tau,\lambda_1,\lambda_2,\eta_1,\eta_2))\,\mathrm{d}s = \eta_2 - \lambda_2 + 0.1
\end{aligned} \tag{90}$$

together with equations (88), (89). System (90), (88), (89) consists of seven equations and has to be solved numerically with respect to the same number of unknowns τ, ξ_1, ξ_2, λ_1, λ_2, η_1, and η_2. Note that this task is considerably simpler than solving (61) with $m = 2$.

Table 3
Approximate values of parameters for the second solution of problem (70)–(72).

Variable	$m = 1$	$m = 2$	$m = 3$	$m = 4$
τ	0.181450919	0.181450845	0.181450845	0.181450846
ξ_1	−0.492769263	−0.492769235	−0.492769235	−0.492769235
ξ_2	0.003615368	0.003615383	0.003615383	0.003615383
λ_1	−0.491120618	−0.491120590	−0.491120590	−0.491120590
λ_2	0.039921157	0.039921156	0.039921156	0.039921156
η_1	0.007600000	0.007600000	0.007600000	0.007600000
η_2	0.010065508	0.010065542	0.010065542	0.010065542

Finding the roots of (90), (88), (89), we obtain the values presented in the second column of Table 2, and by inserting them into the expressions for $\hat{x}_2(\cdot\,;\tau,\xi,\lambda)$ and $\hat{y}_2(\cdot\,;\tau,\lambda,\eta)$, we obtain the functions

$$\begin{aligned}
X_{21}(t) &= -8.43747133 + 15.75100318\,t + 0.04961612083\,t^2 + 0.2650485212\,t^3\\
&\quad -1.111749125\,t^4 + 0.1405463792\,t^5 - 0.08702093740\,t^6\\
&\quad +0.03984063677\,t^7 - 0.5738497521\cdot 10^{-3}t^8 + 0.213241475\cdot 10^{-5}t^9,\\
X_{22}(t) &= -3.968735665 + 0.1999734728\,t - 0.4960959774\,t^2 + 0.5224312056\,t^3\\
&\quad +0.01398407462\,t^4 - 0.0364420242\,t^5 + 0.878480955\cdot 10^{-2}t^6 - 0.6258338571\cdot 10^{-4}t^7,\\
Y_{21}(t) &= -8.241206558 + 16.58434465\,t + 0.01040592499\,t^2 + 0.1930215965\,t^3\\
&\quad -1.175096781\,t^4 + 0.1400768463\,t^5 - 0.08715111796\,t^6\\
&\quad +0.04289976329\,t^7 - 0.568716202\cdot 10^{-3}t^8 + 0.1940602001\cdot 10^{-5}t^9,\\
Y_{22}(t) &= -4.072094611 + 0.2000007263\,t - 0.4846318681\,t^2 + 0.5552588696\,t^3\\
&\quad +0.01178866493\,t^4 - 0.03702649888\,t^5 + 0.912433001\cdot 10^{-2}t^6 - 0.5970235357\cdot 10^{-4}t^7,
\end{aligned}$$

which are used for the computation of the third iteration.

6.5. Higher iterations

Higher iterations are constructed by analogy. For $m = 3$ and $m = 4$, the corresponding values of parameters obtained by solving the approximate determining equations are shown, respectively, in the third and fourth columns of Table 2. Carrying out computations in Maple and constructing the corresponding functions $X_4 = \mathrm{col}(X_{41}, X_{42})$ and $Y_4 = \mathrm{col}(Y_{41}, Y_{42})$, we show that condition (69) holds for $m = 4$. More precisely, for $\hat{\tau} \approx 0.377366355$ and every $t \in (\hat{\tau}, 0.5]$, the value of $(Y_{41}(t) + 1/2)^2 + Y_{42}(t) - 1/25$ is strictly negative and belongs to the interval $[-4, -1]$. Consequently, the function

$$\hat{u}(t) = \begin{cases} X_4(t) & \text{if } t \in \left[0, \hat{\tau}\right],\\ Y_4(t) & \text{if } t \in \left(\hat{\tau}, \tfrac{1}{2}\right], \end{cases}$$

is the fourth approximation of the first solution of problem (70)–(72). The graph of the solution is shown on Fig. 1a and, componentwise, on Fig. 1b and c. Fig. 2 shows barrier (73) and the point where it is intersected by the solution.

Substituting the approximation $\hat{u}$ of the first solution into system (70), we obtain a residual estimated as follows:

$$\max_{t\in[0,\hat{\tau}]}\left|X_{41}'(t) - X_{42}^2(t) + \frac{t}{5}X_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25}\right| \approx 1.1\cdot 10^{-7},$$

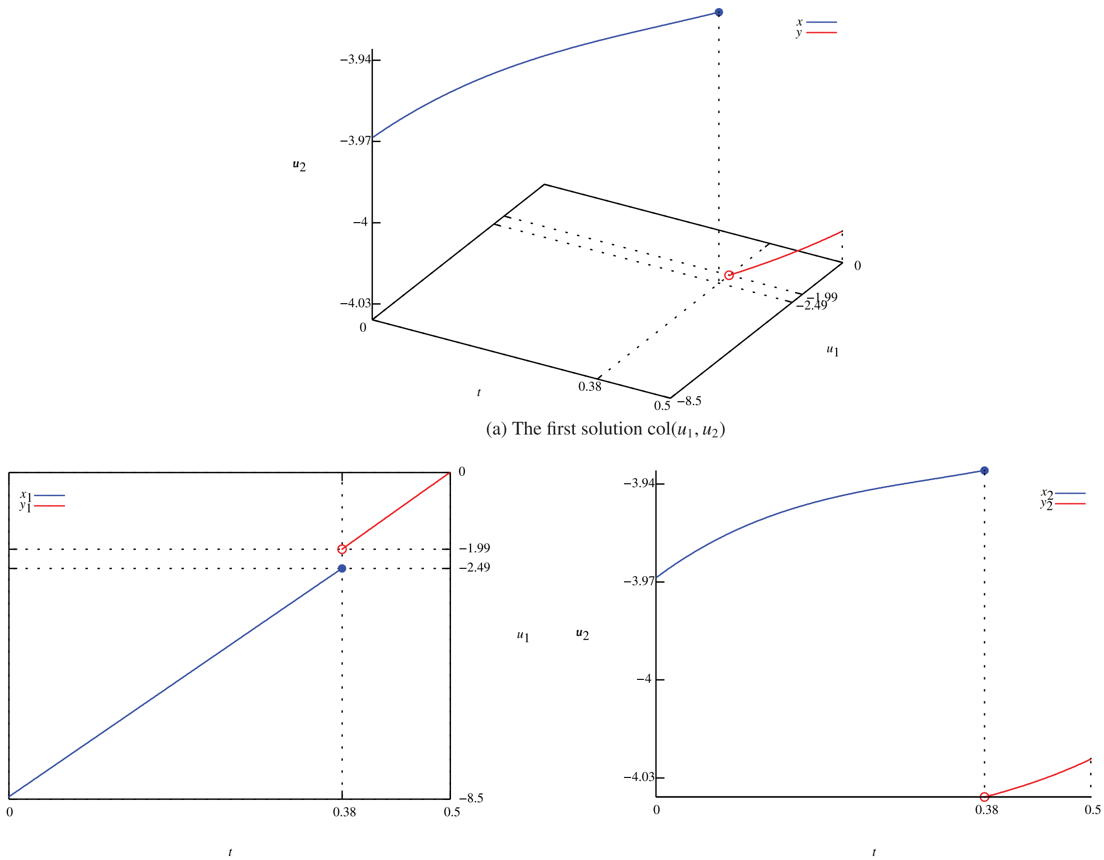
$$\max_{t\in[0,\hat{\tau}]}\left|X_{42}'(t) - \frac{t^2}{10}X_{42}(t) - \frac{t}{8}X_{41}(t) + \frac{21}{800}t^3 - \frac{1}{16}t - \frac{1}{5}\right| \approx 3.1\cdot 10^{-8},$$

$$\max_{t\in[\hat{\tau},\frac{1}{2}]}\left|Y_{41}'(t) - Y_{42}^2(t) + \frac{t}{5}Y_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25}\right| \approx 4.0\cdot 10^{-8},$$

$$\max_{t\in[\hat{\tau},\frac{1}{2}]}\left|Y_{42}'(t) - \frac{t^2}{10}Y_{42}(t) - \frac{t}{8}Y_{41}(t) + \frac{21}{800}t^3 - \frac{1}{16}t - \frac{1}{5}\right| \approx 6.6\cdot 10^{-9}.$$

6.6. Detection of the second solution

Further Maple computations show that, for $\xi_1 \in [-1, 0]$, system (84)–(89) has another root, which leads us to another solution of problem (70)–(72). The approximate values of parameters and functions X_m and Y_m for $1 \le m \le 4$ are found similarly to the preceding case. The numerical values of parameters are shown in Table 3. One can also verify that inequality (69) holds for



(a) The first solution $\text{col}(u_1, u_2)$

(b) Component u_1

(c) Component u_2

Fig. 1. The first solution of problem (70)–(72).

$m = 4$. More precisely, for $\hat{\tau} \approx 0.1814508455$ and any $t \in (\hat{\tau}, 0.5]$, the value of $(Y_{41}(t) + 1/2)^2 + Y_{42}(t) - 1/25$ is strictly positive and belongs to the interval [0.16, 0.23]. Consequently, the function

$$\hat{u}(t) = \begin{cases} X_4(t) & \text{if } t \in [0, \hat{\tau}], \\ Y_4(t) & \text{if } t \in \left(\hat{\tau}, \frac{1}{2}\right], \end{cases}$$

is the fourth approximation of the second solution of problem (70)–(72). Fig. 3 shows barrier (73) and its intersection point with the second solution.

The accuracy of approximation of the second solution can be examined by substituting $\hat{u}$ into (70), which produces a residual estimated as follows:

$$\max_{t \in [0, \hat{\tau}]} \left| X_{41}'(t) - X_{42}^2(t) + \frac{t}{5} X_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| \approx 3 \cdot 10^{-11},$$

$$\max_{t \in [0, \hat{\tau}]} \left| X_{42}'(t) - \frac{t^2}{10} X_{42}(t) - \frac{t}{8} X_{41}(t) + \frac{21}{800} t^3 - \frac{t}{16} - \frac{1}{5} \right| \approx 4 \cdot 10^{-12},$$

$$\max_{t \in [\hat{\tau}, \frac{1}{2}]} \left| Y_{41}'(t) - Y_{42}^2(t) + \frac{t}{5} Y_{41}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| \approx 1 \cdot 10^{-10},$$

$$\max_{t \in [\hat{\tau}, \frac{1}{2}]} \left| Y_{42}'(t) - \frac{t^2}{10} Y_{42}(t) - \frac{t}{8} Y_{41}(t) + \frac{21}{800} t^3 - \frac{t}{16} - \frac{1}{5} \right| \approx 1.15 \cdot 10^{-10}.$$

6.7. Testing other barriers

A similar argument is applicable if barriers different from (73) are considered. In particular, we have carried out our computations for problem (70)–(72) with the jump

$$\gamma = \text{col}(0.55, -0.15)$$

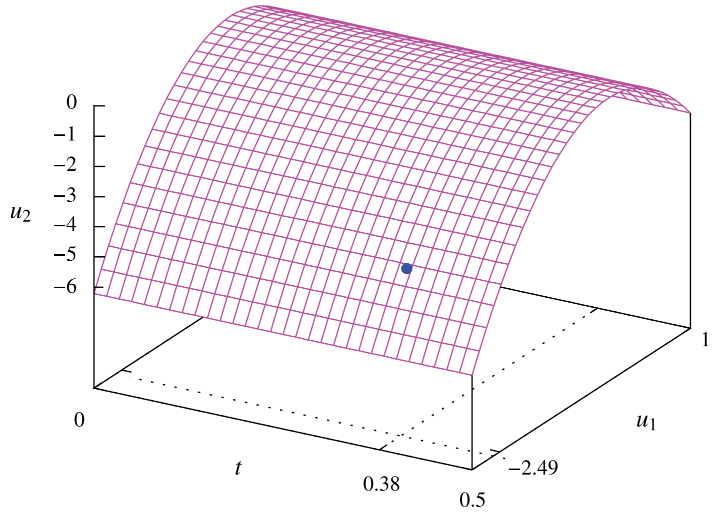


Fig. 2. Barrier (73) and its intersection point $(\tau, u_1(\tau), u_2(\tau))$ with the first solution of problem (70)–(72).

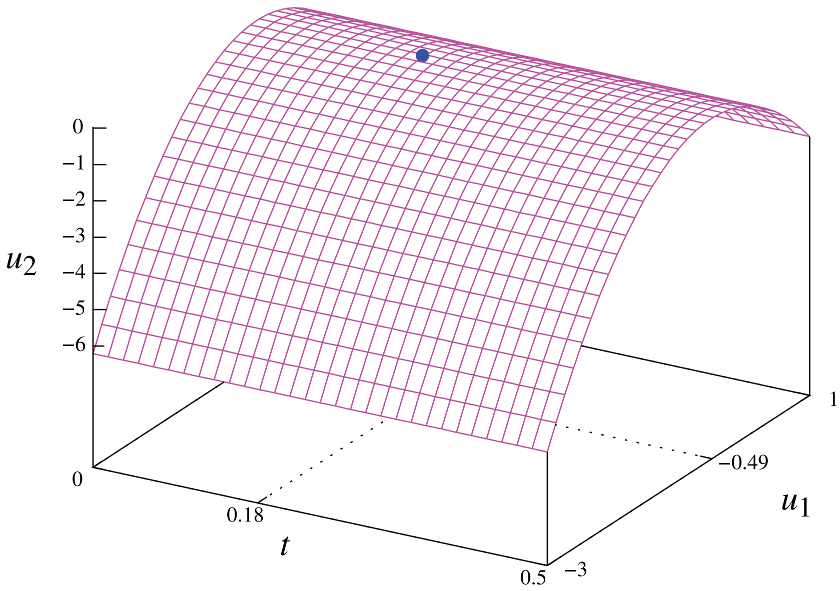


Fig. 3. Barrier (73) and its intersection point with the second solution of problem (70)–(72) with the jump $\gamma = \text{col}(0.55, -0.15)$.

and one of the following three barriers:

$$G_1 := \{(t, x_1, x_2) : x_1^2 + x_2^2 - t = 0,\ 0 \le t \le 1/2\}, \tag{91}$$

$$G_2 := \{(t, x_1, x_2) : x_1^2 + x_2 - t = 0,\ 0 \le t \le 1/2\}, \tag{92}$$

$$G_3 := \{(t, x_1, x_2) : (x_1 + 1/2)^2 + t^2 - 1/10 = 0,\ 0 \le t \le 1/2\}. \tag{93}$$

In all the three cases (91)–(93), the third and fourth approximations of a solution are very close to one another and inequality (69) holds for $m = 4$. The intersection points of the solution of problem (70)–(72) with G replaced by G_i, $i = 1, 2, 3$, given by (91)–(93) are shown, respectively, on Fig. 4a–c.

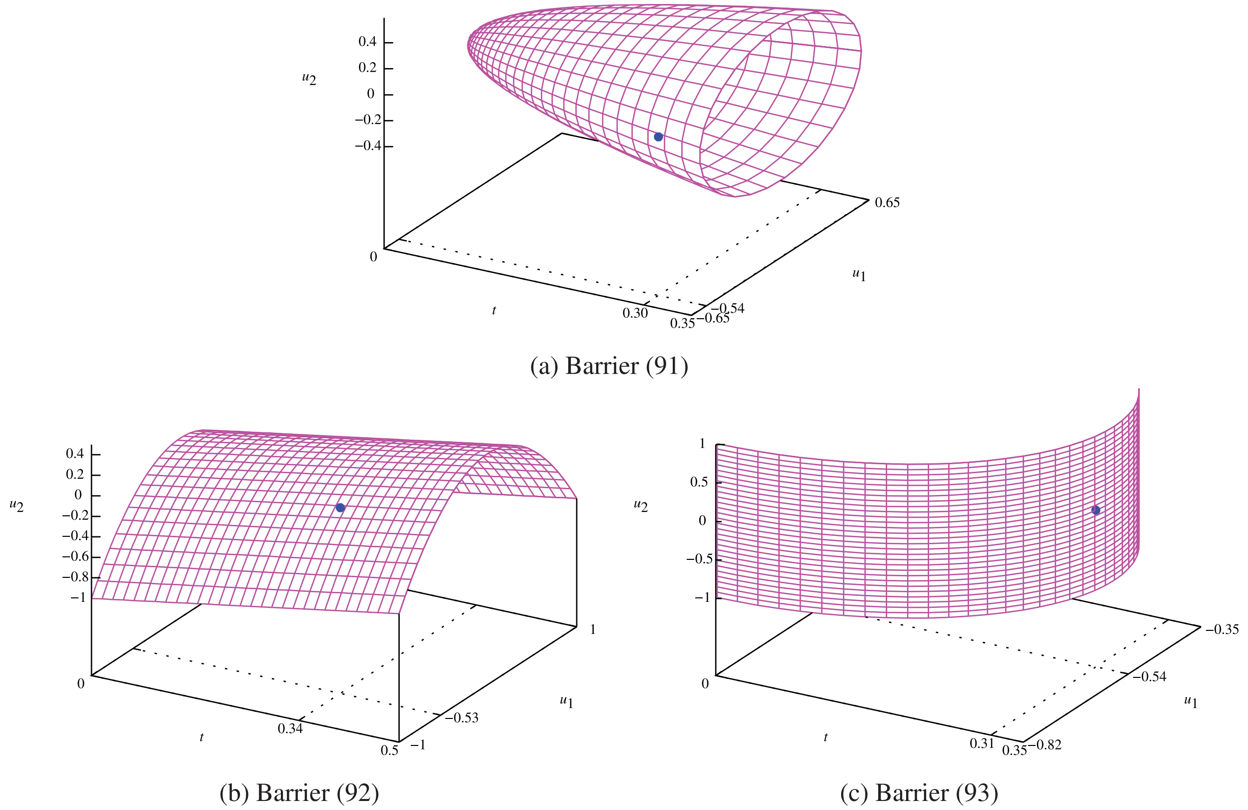


Fig. 4. Barriers and the points where they are intersected by the solution of problem (70)–(72) with the jump $\gamma = \text{col}(0.55, -0.15)$.

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Research Article

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Investigation of solutions of state-dependent multi-impulsive boundary value problems

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Abstract: We describe a reduction technique allowing one to combine an analysis of the existence of solutions with an efficient construction of approximate solutions for a state-dependent multi-impulsive boundary value problem which consists of non-linear system of differential equations

$$u'(t) = f(t, u(t)) \quad \text{for a.e. } t \in [a, b],$$

subject to the state-dependent impulse condition

$$u(t+) - u(t-) = \gamma_t(u(t-)) \quad \text{for } t \in (a, b) \text{ such that } g(t, u(t-)) = 0,$$

and the non-linear two-point boundary condition

$$V(u(a), u(b)) = 0.$$

Keywords: State-dependent multi-impulsive systems, non-linear boundary value problem, parametrization technique, successive approximations

MSC 2010: 34B37, 34B15

Dedicated to Professor Ivan Kiguradze on the occasion of his eightieth birthday

1 Problem setting

We consider the non-linear system of differential equations

$$\frac{du(t)}{dt} = f(t, u(t)) \quad \text{for a.e. } t \in [a, b] \tag{1.1}$$

with $-\infty < a < b < \infty$ and a continuous vector function $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$. Equation (1.1) is subject to the non-linear two-point boundary condition

$$V(u(a), u(b)) = 0, \tag{1.2}$$

where $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and the jump condition

$$u(t+) - u(t-) = \gamma_t(u(t-)) \quad \text{for } t \in (a, b) \text{ such that } g(t, u(t-)) = 0. \tag{1.3}$$

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Here, the functions $\gamma_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in (a, b)$, and $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ are continuous. The set

$$G := \{(t, x) \in [a, b] \times \mathbb{R}^n : g(t, x) = 0\} \tag{1.4}$$

determined by the function g from (1.3) contains all the points of the phase space where the jumps occur and is called a *barrier*.

The time instants t involved in condition (1.3) depend on a solution u through the equation $g(t, u(t-)) = 0$ and are a priori unknown. The jumps of trajectories that may occur in this system are thus *state-dependent*.

Differential models involving state-dependent jumps arise in a number of real life problems (see, e.g., [1–4, 13, 14]). It should be noted, however, that the majority of results available at the moment concern systems with impulses occurring at fixed times. This is due to the fact that the presence of state-dependent jumps significantly changes the properties of boundary value problems. We refer to [6, 7] for more details, where new existence theorems for boundary value problems with both state-dependent and fixed-time jumps are proven. The results available for boundary value problems with state-dependent jumps concern mostly the case of linear boundary conditions and barriers of the form

$$\{(t, x) \in [a, b] \times \mathbb{R}^n : t = g(x)\}, \tag{1.5}$$

which are the special cases of (1.2) and (1.4), respectively. Only solutions intersecting the barrier set exactly once are treated in the papers cited. Moreover, according to our knowledge, *no numerical results* for boundary value problems with state-dependent impulses are currently available in the literature, except of [5]. In particular, the paper [5] provides a way to detect a solution of a problem of this kind in the case where the boundary conditions are linear and the solution intersects the barrier only once.

Here, we study solutions of problem (1.1), (1.2), (1.3) that are allowed to meet the barrier *finitely many* times. More precisely, we study solutions with p jumps defined as follows.

Definition 1.1. Let $p \in \mathbb{N}$. A left continuous vector function $u : [a, b] \to \mathbb{R}^n$ is called a *solution* of problem (1.1), (1.2), (1.3) *with p jumps* if (1.2) holds and there exist points $a < \tau_1 < \tau_2 < \cdots < \tau_p < b$ such that
(1) the restrictions $u|_{[a,\tau_1]}, u|_{(\tau_1,\tau_2]}, \dots, u|_{(\tau_p,b]}$ have continuous derivatives,
(2) u satisfies (1.1) for $t \in [a, b] \setminus \{\tau_1, \tau_2, \dots, \tau_p\}$,
(3) the conditions

$$g(\tau_i, u(\tau_i)) = 0, \quad i = 1, \dots, p, \qquad g(t, u(t)) \neq 0, \quad t \in [a, b] \setminus \{\tau_1, \dots, \tau_p\}, \tag{1.6}$$

$$u(\tau_i+) - u(\tau_i) = \gamma_{\tau_i}(u(\tau_i)), \quad i = 1, \dots, p, \tag{1.7}$$

hold.

Note that both differential equation (1.1) and boundary conditions (1.2) are, generally speaking, non-linear, and, in contrast to (1.5), appearing in the earlier works, a barrier set of the general form (1.4) is considered.

2 Notation and subsidiary statements

(1) The notation $|x| = \mathrm{col}(|x_1|, \dots, |x_n|)$ is used for any $x = \mathrm{col}(x_1, \dots, x_n) \in \mathbb{R}^n$. The operations "max" and "min" for vector functions as well as the inequalities between vectors are understood componentwise.
(2) 1_n is the unit matrix of dimension n.
(3) 0_n is the zero matrix of dimension n.
(4) $r(K)$ is the maximal, in modulus, eigenvalue of a matrix K.
(5) For $D \subset \mathbb{R}^n$ and $f : [a, b] \times D \to \mathbb{R}^n$, the notation $f \in \mathrm{Lip}(K, D)$ means that there exists a square matrix K with non-negative entries satisfying the *componentwise Lipschitz condition*

$$|f(t, u_1) - f(t, u_2)| \le K|u_1 - u_2|, \quad t \in [a, b],\ u_1, u_2 \in D.$$

(6) For a non-negative vector $\varrho \in \mathbb{R}^n$, a *componentwise* ϱ-neighbourhood of a point $z \in \mathbb{R}^n$ is defined as

$$B(z, \varrho) := \{v \in \mathbb{R}^n : |v - z| \le \varrho\}. \tag{2.1}$$

(7) For given two sets $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^n$ we introduce the set

$$\mathcal{H}(D_1, D_2) := \{(1-\theta)z_1 + \theta z_2 : z_1 \in D_1, z_2 \in D_2,\ \theta \in [0,1]\}. \tag{2.2}$$

(8) If $D \subset \mathbb{R}^n$ is a compact set and $f : [a,b] \times D \to \mathbb{R}^n$ is continuous, we put

$$\delta_f(D) := \max_{(t,x)\in[a,b]\times D} f(t,x) - \min_{(t,x)\in[a,b]\times D} f(t,x). \tag{2.3}$$

(9) Let $-\infty < t_0 < t_1 < \infty$ be arbitrary and

$$\alpha_1(t; t_0, t_1) = 2(t-t_0)\Big(1 - \frac{t-t_0}{t_1-t_0}\Big), \quad t \in [t_0, t_1]. \tag{2.4}$$

It follows immediately from (2.4) that $\alpha(\cdot\,; t_0, t_1)$ is non-negative and

$$\max_{t\in[t_0,t_1]} \alpha_1(t; t_0, t_1) = \frac{1}{2}(t_1 - t_0). \tag{2.5}$$

Function (2.4) is useful in view of the following lemma.

Lemma 2.1 ([8, Lemma 3.13]). *The estimate*

$$\left| \int_{t_0}^{t} z(s)\,\mathrm{d}s - \frac{t-t_0}{t_1-t_0} \int_{t_0}^{t_1} z(s)\,\mathrm{d}s \right| \le \frac{1}{2}\alpha_1(t; t_0, t_1)\Big(\max_{s\in[t_0,t_1]} z(s) - \min_{s\in[t_0,t_1]} z(s)\Big)$$

holds for arbitrary $t \in [t_0, t_1]$ and continuous vector function $z : [t_0, t_1] \to \mathbb{R}^n$.

Let the functions $\{\alpha_m(\cdot\,; t_0, t_1) : m \ge 1\}$ be defined by the recurrence relation

$$\alpha_{m+1}(t; t_0, t_1) = (1 - \frac{t-t_0}{t_1-t_0}) \int_{t_0}^{t} \alpha_m(s; t_0, t_1)\,\mathrm{d}s + \frac{t-t_0}{t_1-t_0} \int_{t}^{t_1} \alpha_m(s; t_0, t_1)\,\mathrm{d}s, \quad t \in [t_0, t_1], \tag{2.6}$$

where $m = 0, 1, \dots$ and $\alpha_0(\cdot\,; t_0, t_1) \equiv 1$. For $m = 0$, formula (2.6) reduces to (2.4).

Lemma 2.2 ([8, Lemma 3.16]). *The estimate*

$$\alpha_{m+1}(t; t_0, t_1) \le \frac{10}{9}\Big(\frac{3(t_1-t_0)}{10}\Big)^m \alpha_1(t; t_0, t_1), \quad t \in [t_0, t_1],$$

holds for any $m = 0, 1, \dots$.

3 Sets of parameters and auxiliary model problems

Let $p \in \mathbb{N}$. The idea that we are going to follow is to approximate a solution u of problem (1.1), (1.2), (1.3) with p jumps (see Definition 1.1) by suitable sequences of functions *separately* on the interval $[a, \tau_1]$ preceding the moments $\tau_1, \tau_2, \dots, \tau_p$ of jumps (*pre-jump* evolution) and then sequentially on the intervals

$$[\tau_1, \tau_2], [\tau_2, \tau_3], \dots, [\tau_{p-1}, \tau_p], [\tau_p, \tau_{p+1}], \tag{3.1}$$

with $\tau_{p+1} := b$, which correspond to the *after-jump* evolution. The time instants where the jumps occur are treated as the parameters to be determined later. The key role in our analysis will be played by the values $\tau_1, \tau_2, \dots, \tau_p$ and $\lambda^{[1]}, \lambda^{[2]}, \dots, \lambda^{[p]}$, representing, respectively, the unknown jump times and pre-jump values of the solution, and ξ, $\lambda^{[p+1]}$ representing the values of the solution at the points a and b.

Consider $(p+2)$ compact sets

$$\Omega_0, \Omega_1, \dots, \Omega_p, \Omega_{p+1} \subset \mathbb{R}^n, \tag{3.2}$$

and, applying a shift by γ_{τ_k} from (1.7), define the sets

$$\Omega_k^+ := \{x + \gamma_{\tau_k}(x) : x \in \Omega_k\}, \quad k = 1, \dots, p.$$

Let us focus on solutions u of problem (1.1), (1.2), (1.3) with p jumps such that

$$u(a) \in \Omega_0, \quad u(\tau_{p+1}) \in \Omega_{p+1},$$
$$u(\tau_k) \in \Omega_k, \quad u(\tau_k+) \in \Omega_k^+, \quad k = 1, \dots, p.$$

The techniques to be applied will require to define suitable neighbourhoods of sets where the values of parameters are looked for. For this purpose, we choose some vectors $\varrho^{[0]}, \varrho^{[1]}, \varrho^{[2]}, \dots, \varrho^{[p]}$ from $\mathbb{R}^n$ and, using notation (2.1), (2.2), construct the sets

$$\mathcal{U}_0 := \bigcup_{v \in \mathcal{H}(\Omega_0, \Omega_1)} B(v, \varrho^{[0]}), \tag{3.3}$$

$$\mathcal{U}_k := \bigcup_{v \in \mathcal{H}(\Omega_k^+, \Omega_{k+1})} B(v, \varrho^{[k]}), \quad k = 1, \dots, p. \tag{3.4}$$

Then, in Section 4.1, an auxiliary two-point boundary value problem

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \in [a, \tau_1], \qquad x(a) = \xi, \quad x(\tau_1) = \lambda^{[1]}, \tag{3.5}$$

is studied, where ξ and $\lambda^{[1]}$ are treated as free parameters belonging, respectively, to Ω_0 and Ω_1. Further on, in Section 4.2, we consider p auxiliary parametrized two-point boundary value problems

$$\begin{gathered} \frac{dy^{[k]}(t)}{dt} = f(t, y^{[k]}(t)), \quad t \in [\tau_k, \tau_{k+1}], \\ y^{[k]}(\tau_k) = \lambda^{[k]} + \gamma_k(\lambda^{[k]}), \quad y^{[k]}(\tau_{k+1}) = \lambda^{[k+1]}, \quad k = 1, 2, \dots, p. \end{gathered} \tag{3.6}$$

The variables

$$\begin{gathered} a < \tau_1 < \dots < \tau_p < \tau_{p+1} = b, \quad \xi = \operatorname{col}(\xi_1, \xi_2, \dots, \xi_n) \in \Omega_0, \\ \lambda^{[k]} = \operatorname{col}(\lambda_1^{[k]}, \lambda_2^{[k]}, \dots, \lambda_n^{[k]}) \in \Omega_k, \quad k = 1, 2, \dots, p+1, \end{gathered} \tag{3.7}$$

appearing in (3.5) and (3.6) are considered as unknown parameters. We are going to consider solutions $x(\cdot)$ and $y^{[k]}(\cdot)$, $k = 1, \dots, p$, of the auxiliary problems (3.5) and (3.6) with range in the sets $\mathcal{U}_0$ and $\mathcal{U}_k$ (see (3.3) and (3.4)). It should be noted that, although both (3.5) and (3.6) are formally overdetermined problems (with n equations and $2n$ boundary conditions), one can see that, in fact, due to the nature of the boundary conditions imposed, no complications arise in this relation when (3.5) and (3.6) are treated together.

We shall see below that the families of simpler problems (3.5) and (3.6) can be efficiently used in a constructive analysis of the original problem (1.1), (1.2), (1.3).

4 Construction of iterations

As has already been mentioned, approximations are constructed separately on the interval $[a, \tau_1]$, where still no jumps occur, and each of the p intervals (3.1) corresponding to the system evolution between the jumps.

4.1 Iterations for the first pre-jump evolution

To study problem (3.5) on $[a, \tau_1] \times \mathcal{U}_0$, we introduce the parameterized sequences of vector functions

$$x_m(t) := x_m(t; \tau_1, \xi, \lambda^{[1]}), \quad t \in [a, \tau_1], \tag{4.1}$$

with the parameters $\tau_1, \xi, \lambda^{[1]}$ from (3.7), by the relations

$$x_0(t) = \xi + \frac{t-a}{\tau_1 - a}(\lambda^{[1]} - \xi), \tag{4.2}$$

$$x_m(t) = x_0(t) + \int_a^t f(s, x_{m-1}(s))\,ds - \frac{t-a}{\tau_1 - a}\int_a^{\tau_1} f(s, x_{m-1}(s))\,ds, \quad t \in [a, \tau_1],\ m \in \mathbb{N}. \tag{4.3}$$

Here and below, in order not to overload the notation, we often do not specify explicitly the dependence on the corresponding parameters in longer formulas and keep in braces only the time variable.

Clearly, (4.2) can be rewritten as

$$x_0(t) = \Big(1 - \frac{t-a}{\tau_1 - a}\Big)\xi + \frac{t-a}{\tau_1 - a}\lambda^{[1]}, \quad t \in [a, \tau_1],$$

and, therefore, equalities (4.2), (4.3) imply immediately the following proposition.

Proposition 4.1. *The equalities*

$$x_m(a) = \xi, \quad x_m(\tau_1) = \lambda^{[1]}$$

are true for any $m \geq 0$.

The following statement establishes the uniform convergence of sequence (4.3).

Theorem 4.2 (Uniform convergence). *Assume that there exist a non-negative vector* $\varrho^{[0]}$ *and a matrix* K_0 *such that*

$$\varrho^{[0]} \geq \frac{b-a}{4}\delta_f(\mathcal{U}_0), \quad r(K_0) < \frac{10}{3(b-a)}, \tag{4.4}$$

where $\mathcal{U}_0$ *and* $\delta_f(\mathcal{U}_0)$ *are defined according to* (3.3) *and* (2.3). *In addition, let* f *satisfy the condition*

$$f \in \operatorname{Lip}(K_0, \mathcal{U}_0). \tag{4.5}$$

Then for arbitrary values of the parameters from (3.7):

(1) *For any* $m \geq 0$, $x_m(\cdot)$ *is continuously differentiable on* $[a, \tau_1]$ *and*

$$\{x_m(t) : t \in [a, \tau_1]\} \subset \mathcal{U}_0. \tag{4.6}$$

(2) *There exists a vector function* $x_\infty : [a, \tau_1] \to \mathcal{U}_0$ *such that*

$$x_\infty(t) = \lim_{m\to\infty} x_m(t) \quad \textit{uniformly in } t \in [a, \tau_1]. \tag{4.7}$$

(3) *The limit* x_∞ *is a unique continuously differentiable solution with values in* $\mathcal{U}_0$ *of the perturbed boundary value problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)) + \frac{1}{\tau_1 - a}\Psi_0, \quad t \in [a, \tau_1], \qquad x(a) = \xi, \quad x(\tau_1) = \lambda^{[1]}, \tag{4.8}$$

where the vector $\Psi_0 \in \mathbb{R}^n$ *depending on the parameters* τ_1, ξ, *and* $\lambda^{[1]}$ *is given by the formula*

$$\Psi_0 := \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, x_\infty(s))\,\mathrm{d}s.$$

(4) *The error estimate*

$$|x_\infty(t) - x_m(t)| \leq \frac{5}{9}\alpha_1(t; a, \tau_1)Q_0^m(1_n - Q_0)^{-1}\delta_f(\mathcal{U}_0)$$

holds on $[a, \tau_1]$ *for any* $m \geq 1$, *where* $\alpha_1(\cdot\,; a, \tau_1)$ *is as in* (2.4) *and*

$$Q_0 = \frac{3(b-a)}{10}K_0. \tag{4.9}$$

Proof. Note that (4.8) is an ordinary differential equation and differs from the original equation (1.1) by the additive constant forcing term $\Psi_0/(\tau_1 - a)$. We will argue as in [5, 8] using Lemmas 2.1 and 2.2. Since $\tau_1 \in (a, b)$, $\xi \in \Omega_0$, and $\lambda^{[1]} \in \Omega_1$, using (2.2), (3.3) and (4.2), we obtain

$$\{x_0(t) : t \in [a, \tau_1]\} \subset \mathcal{H}(\Omega_0, \Omega_1) \subset \mathcal{U}_0.$$

Consider an arbitrary fixed $m \in \mathbb{N} \cup \{0\}$ and assume that

$$\{x_m(t) : t \in [a, \tau_1]\} \subset \mathfrak{U}_0.$$

Then, by virtue of Lemma 2.1 and relations (2.3), (2.5), (4.3) and (4.4), the estimate

$$|x_{m+1}(t) - x_0(t)| \le \frac{1}{2}\alpha_1(t; a, \tau_1)\delta_f(\mathfrak{U}_0) \le \varrho^{[0]}, \quad t \in [a, \tau_1], \tag{4.10}$$

holds, whence, in view of (3.3), it follows that

$$\{x_{m+1}(t) : t \in [a, \tau_1]\} \subset \mathfrak{U}_0.$$

This means that (4.10) and (4.6) hold for every $m \ge 0$. Inequality (4.10) with $m = 0$ implies that

$$|x_1(t) - x_0(t)| \le \frac{1}{2}\alpha_1(t; a, \tau_1)\delta_f(\mathfrak{U}_0), \quad t \in [a, \tau_1].$$

Further on, using (2.6), (4.3), (4.5) and (4.10), we get

$$\begin{aligned} |x_2(t) - x_1(t)| &\le K_0\Bigg(\Big(1 - \frac{t-a}{\tau_1 - a}\Big)\int_a^t |x_1(s) - x_0(s)|\,ds + \frac{t-a}{\tau_1 - a}\int_t^{\tau_1} |x_1(s) - x_0(s)|\,ds\Bigg) \\ &\le \frac{1}{2}K_0\delta_f(\mathfrak{U}_0)\alpha_2(t; a, \tau_1), \quad t \in [a, \tau_1], \end{aligned}$$

and, by induction,

$$|x_{m+1}(t) - x_m(t)| \le \frac{1}{2}K_0^m\delta_f(\mathfrak{U}_0)\alpha_{m+1}(t; a, \tau_1), \quad t \in [a, \tau_1],\ m \in \mathbb{N}.$$

Therefore, using Lemma 2.2, we obtain

$$\begin{aligned} |x_{m+1}(t) - x_m(t)| &\le \frac{1}{2}K_0^m\delta_f(\mathfrak{U}_0)\frac{10}{9}\Big(\frac{3(b-a)}{10}\Big)^m\alpha_1(t; a, \tau_1) \\ &= \frac{5}{9}Q_0^m\,\delta_f(\mathfrak{U}_0)\,\alpha_1(t; a, \tau_1) \end{aligned} \tag{4.11}$$

for all $t \in [a, \tau_1]$, $m \in \mathbb{N}$. Consider arbitrary $m, j \in \mathbb{N}$. In view of (4.4) and (4.9), it follows that $r(Q_0) < 1$. Then (4.11) yields

$$\begin{aligned} |x_{m+j}(t) - x_m(t)| &\le \frac{5}{9}\alpha_1(t; a, \tau_1)Q_0^m\sum_{i=0}^{j-1}Q_0^i\delta_f(\mathfrak{U}_0) \\ &\le \frac{5}{9}\alpha_1(t; a, \tau_1)Q_0^m(1_n - Q_0)^{-1}\delta_f(\mathfrak{U}_0), \quad t \in [a, \tau_1], \end{aligned}$$

and the remaining assertions of Theorem 4.2 follow. □

4.2 Iterations for evolution with jumps

Let $k \in \{1, \dots, p\}$. To study problem (3.6) on $[\tau_k, \tau_{k+1}] \times \mathfrak{U}_k$, we introduce the parameterized sequence of vector functions

$$y_m^{[k]}(t) := y_m^{[k]}(t; \tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}), \quad t \in [\tau_k, \tau_{k+1}],$$

by the relations

$$y_0^{[k]}(t) = (\lambda^{[k]} + \gamma_{\tau_k}(\lambda^{[k]})) + \frac{t - \tau_k}{\tau_{k+1} - \tau_k}(\lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]})), \tag{4.12}$$

$$y_m^{[k]}(t) = y_0^{[k]}(t) + \int_{\tau_k}^t f(s, y_{m-1}^{[k]}(s))\,ds - \frac{t - \tau_k}{\tau_{k+1} - \tau_k}\int_{\tau_k}^{\tau_{k+1}} f(s, y_{m-1}^{[k]}(s))\,ds, \quad m \in \mathbb{N}. \tag{4.13}$$

The parameters $\tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}$ here are as in (3.7), Similarly to Proposition 4.1 for sequence (4.1), we see that (4.12) can be rewritten as

$$y_0^{[k]}(t) = \left(1 - \frac{t-\tau_k}{\tau_{k+1}-\tau_k}\right)(\lambda^{[k]} + \gamma_{\tau_k}(\lambda^{[k]})) + \frac{t-\tau_k}{\tau_{k+1}-\tau_k}\lambda^{[k+1]} \tag{4.14}$$

and, therefore, (4.12) and (4.13) imply immediately:

Proposition 4.3. *The equalities*

$$y_m^{[k]}(\tau_k) = \lambda^{[k]} + \gamma_{\tau_k}(\lambda^{[k]}), \quad y_m^{[k]}(\tau_{k+1}) = \lambda^{[k+1]}$$

hold for any $m \geq 0$.

Theorem 4.4 (Uniform convergence). *Assume that, for each $k \in \{1, 2, \dots, p\}$, there exist a non-negative vector $\varrho^{[k]}$ and a matrix K_k such that*

$$\varrho^{[k]} \geq \frac{b-a}{2}\delta_f(\mathcal{U}_k), \quad r(K_k) < \frac{10}{3(b-a)}, \tag{4.15}$$

where $\mathcal{U}_k$ and $\delta_f(\mathcal{U}_k)$ are defined according to (3.4) and (2.3). Additionally, let f satisfy the condition

$$f \in \mathrm{Lip}(K_k, \mathcal{U}_k). \tag{4.16}$$

Then for all $k \in \{1, \dots, p\}$ and the parameters from (3.7):

(1) *For any $m \geq 0$, the vector function (4.13) is continuously differentiable on $[\tau_k, \tau_{k+1}]$ and*

$$\{y_m^{[k]}(t) : t \in [\tau_k, \tau_{k+1}]\} \subset \mathcal{U}_k.$$

(2) *There exists a vector function $y_\infty^{[k]} : [\tau_k, \tau_{k+1}] \to \mathcal{U}_k$ such that*

$$y_\infty^{[k]}(t) = \lim_{m\to\infty} y_m^{[k]}(t) \quad \text{uniformly on } [\tau_k, \tau_{k+1}]. \tag{4.17}$$

(3) *The limit $y_\infty^{[k]}$ is a unique continuously differentiable solution with values in $\mathcal{U}_k$ of the perturbed boundary value problem on $[\tau_k, \tau_{k+1}]$*

$$\frac{dy^{[k]}(t)}{dt} = f(t, y^{[k]}(t)) + \frac{1}{\tau_{k+1}-\tau_k}\Psi_k, \tag{4.18}$$

$$y^{[k]}(\tau_k) = \lambda^{[k]} + \gamma_{\tau_k}(\lambda^{[k]}), \quad y^{[k]}(\tau_{k+1}) = \lambda^{[k+1]}, \tag{4.19}$$

where $\Psi_k \in \mathbb{R}^n$ depending on the parameters $\tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}$ is given by the formula

$$\Psi_k := \lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y_\infty^{[k]}(s))\, ds.$$

(4) *The error estimate*

$$|y_\infty^{[k]}(t) - y_m^{[k]}(t)| \leq \frac{5}{9}\alpha_1(t; \tau_k, \tau_{k+1}) Q_k^m (1_n - Q_k)^{-1}\delta_f(\mathcal{U}_k)$$

holds on $[\tau_k, \tau_{k+1}]$ for all $m \geq 1$, where

$$Q_k = \frac{3(b-a)}{10}K_k.$$

Proof. It is sufficient to note that (4.18) is an ordinary differential equation which differs from the original equation (1.1) by an additive constant forcing term of the form $\Psi_k/(\tau_{k+1}-\tau_k)$. The required assertions are thus obtained by analogy with the proof of Theorem 4.2. □

Under the conditions of the above theorems, for $m \geq 0$ we can construct the vector functions

$$u_m(t) := \begin{cases} x_m(t) & \text{if } t \in [a, \tau_1], \\ y_m^{[k]}(t) & \text{if } t \in (\tau_k, \tau_{k+1}],\ k = 1, 2, \dots, p, \end{cases}$$

and consider their limit

$$u_\infty(t) := \begin{cases} x_\infty(t) & \text{if } t \in [a, \tau_1], \\ y_\infty^{[k]}(t) & \text{if } t \in (\tau_k, \tau_{k+1}],\ k = 1, 2, \dots, p. \end{cases} \tag{4.20}$$

Then

$$u_\infty : [a, b] \to \mathcal{U}_0 \cup \mathcal{U}_1 \cup \dots \cup \mathcal{U}_p \subset \mathbb{R}^n$$

is a vector function depending on the parameters $\tau_1, \dots, \tau_p \in (a, b)$, $\xi \in \Omega_0$, and $\lambda^{[k]} \in \Omega_k$, $k = 1, \dots, p + 1$. For suitable values of these parameters, function (4.20) is a solution of the original boundary value problem (1.1), (1.2), (1.3) with p jumps. The conditions determining the appropriate values of the parameters are specified in the next section.

In the case where conditions (4.4), (4.5) or (4.15), (4.16) are not fulfilled for the considered problem, one can suggest to adjust the interval halving procedure to handle it. For problems without impulses, this technique is described in [9, 11, 12].

5 Determining equations

We note again that equations (4.8) and (4.18) are ordinary differential equations that differ from the original equation (1.1) by constant forcing terms. This simple observation allows us to argue as in [5, 8] and conclude that the limits x_∞ and $y_\infty^{[k]}$, $k = 1, 2, \dots, p$, in Theorems 4.2 and 4.4 are related to the original impulsive boundary value (1.1), (1.2), (1.3) with p jumps in the following way.

Theorem 5.1. *Let the conditions of Theorems 4.2 and 4.4 be fulfilled and let x_∞ and $y_\infty^{[k]}$, $k = 1, \dots, p$, be functions defined in* (4.7) *and* (4.17). *Then the following assertions hold.*

(1) *Assume that the system of algebraic determining equations for unknown parameters $\tau_1, \dots, \tau_p$, ξ, $\lambda^{[k]}$, $k = 1, \dots, p + 1$,*

$$\begin{aligned} \Psi_0 &:= \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, x_\infty(s))\,ds = 0, \\ \Psi_k &:= \lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y_\infty^{[k]}(s))\,ds = 0, \quad k = 1, 2, \dots, p, \\ & g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, 2, \dots, p, \\ & V(\xi, \lambda^{[p+1]}) = 0 \end{aligned} \tag{5.1}$$

has a solution

$$\tau_1^*, \tau_2^*, \dots, \tau_p^*, \xi^*, \lambda^{[k]*}, \quad k = 1, \dots, p + 1, \tag{5.2}$$

where $a < \tau_1^* < \tau_1^* < \dots < \tau_p^* < b$, $\xi^* \in \Omega_0$, $\lambda^{[k]*} \in \Omega_k$, $k = 1, \dots, p + 1$. Let x_∞^* and $y_\infty^{[k]*}$ be the limit functions (4.7) *and* (4.17) *with the values of parameters* (5.2) *and assume that*

$$\begin{aligned} g(t, x_\infty^*(t)) &\neq 0, \quad t \in [a, \tau_1^*), \\ g(t, y_\infty^{[k]*}(t)) &\neq 0, \quad t \in [\tau_k^*, \tau_{k+1}^*),\ k = 1, 2, \dots, p - 1, \\ g(t, y_\infty^{[p]*}(t)) &\neq 0, \quad t \in [\tau_p^*, b]. \end{aligned} \tag{5.3}$$

Then the vector function

$$u_\infty^*(t) := \begin{cases} x_\infty^*(t) & \textit{if } t \in [a, \tau_1^*], \\ y_\infty^{[k]*}(t) & \textit{if } t \in (\tau_k^*, \tau_{k+1}^*],\ k = 1, 2, \dots, p - 1, \\ y_\infty^{[p]*}(t) & \textit{if } t \in (\tau_p^*, b] \end{cases} \tag{5.4}$$

is a solution of the impulsive boundary value problem (1.1), (1.2), (1.3) *with p jumps at the time instants $\tau_1^*, \tau_2^*, \dots, \tau_p^*$.*

(2) *If u is a solution of problem* (1.1), (1.2), (1.3) *with p jumps at the time instants $\tau_1^*, \tau_2^*, \dots, \tau_p^*$ such that*

$$\begin{gathered}\{u(t) : t \in [a, \tau_1^*]\} \subset \mathcal{U}_0, \quad \{u(t) : t \in (\tau_p^*, b]\} \subset \mathcal{U}_p, \\ \{u(t) : t \in (\tau_k^*, \tau_{k+1}^*]\} \subset \mathcal{U}_k, \quad k = 1, 2, \dots, p-1, \\ u(a) \in \Omega_0, \quad u(b) \in \Omega_{p+1}, \qquad u(\tau_k^*) \in \Omega_k, \quad k = 1, \dots, p,\end{gathered} \tag{5.5}$$

then the values $\xi^* = u(a)$, τ_k^*, $\lambda^{[k]*} = u(\tau_k^*)$, $k = 1, \dots, p$, $\lambda^{[p+1]*} = u(b)$ necessarily satisfy the system of determining equations (5.1).

Note that the system of algebraic determining equations (5.1) consists of $(p + 2)n + p$ scalar equations for $(p + 2)n + p$ scalar unknown parameters (3.7). The number of equations thus coincides with the number of unknown parameters involved. Under the conditions of Theorem 5.1, system (5.1) with condition (5.3) allows one to determine all possible solutions u of problem (1.1), (1.2), (1.3) with values satisfying (5.5) and having p jumps. Consequently, the argument based on Theorem 5.1 allows one to deal with multiple solutions of the problem.

5.1 Solutions with different numbers of jumps

In general, problem (1.1), (1.2), (1.3) may have a solution u with p jumps and another solution v with q jumps, where $p \neq q$. Therefore, if we are interested in finding solutions of problem (1.1), (1.2), (1.3) having various number of jumps at their points of intersection with barrier (1.4), we follow these steps:

(1) Choose $p = 1$ and use our scheme with only one possible jump at the point τ_1. Then system (5.1) of $3n + 1$ scalar algebraic equations has the form

$$\Psi_0 = 0, \qquad \Psi_1 = 0, \qquad g(\tau_1, \lambda^{[1]}) = 0, \quad V(\xi, b) = 0. \tag{5.6}$$

(a) If (5.6) does not have solutions, then a solution u of problem (1.1), (1.2), (1.3) satisfying $u(a) \in \Omega_0$ and having one jump does not exist.

(b) Assume that system (5.6) has a solution

$$\tau_1^* \in (a, b), \quad \xi^* \in \Omega_0, \quad \lambda^{[k]*} \in \Omega_k, \quad k = 1, 2, \tag{5.7}$$

and let

$$\begin{aligned} g(t, x_\infty^*(t)) &\neq 0, \quad t \in [a, \tau_1^*), \\ g(t, y_\infty^{[1]*}(t) &\neq 0, \quad t \in [\tau_1^*, b], \end{aligned} \tag{5.8}$$

where x_∞^* and $y_\infty^{[1]*}$ are obtained from (4.7) and (4.17) by substituting there values into (5.7). Then, according to (5.4), we can conclude that the vector function

$$u_\infty^*(t) := \begin{cases} x_\infty^*(t) & \text{if } t \in [a, \tau_1^*], \\ y_\infty^{[1]*}(t) & \text{if } t \in (\tau_1^*, b], \end{cases}$$

is a solution of the impulsive boundary value problem (1.1), (1.2), (1.3) with one jump at the moment τ_1^*.

(c) If (5.8) is not fulfilled, that is either $g(\cdot, x_\infty^*(\cdot))$ has one or more roots in (a, τ_1^*) or $g(\cdot, y_\infty^{[1]*}(\cdot))$ has one or more roots in (τ_1^*, b), then, according to Definition 1.1, the function $u = u_\infty^*$ determined by values (5.7) is not a solution of problem (1.1), (1.2), (1.3) satisfying $u(a) \in \Omega_0$ and having one jump. In that case, other roots of (5.6) should be looked for.

(2) If we do not succeed in looking for a solution with one jump, there is still a possibility that there exists a solution of problem (1.1), (1.2), (1.3) with more intersection points with barrier (1.4) at which the corresponding jumps occur. In order to detect such solutions, we repeatedly apply our scheme for higher values of p. Note that solutions with multiple jumps may exist even in cases where a one-jump solution had been found in Step (1) (see an example in Section 10.1).

(a) For $p = 2$, system (5.1) of $4n + 2$ scalar algebraic equations has the form

$$\Psi_0 = 0, \qquad \Psi_k = 0, \qquad g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, 2, \qquad V(\xi, b) = 0. \tag{5.9}$$

(b) If (5.9) has not a solution, then a solution u of problem (1.1), (1.2), (1.3) satisfying $u(a) \in \Omega_0$ and having two jumps does not exist.

(c) Assume that system (5.9) has a solution

$$\tau_1^*, \tau_2^* \in (a, b), \qquad \xi^* \in \Omega_0, \qquad \lambda^{[k]*} \in \Omega_k, \quad k = 1, 2, 3, \tag{5.10}$$

and let

$$\begin{aligned} g(t, x_\infty^*(t)) &\neq 0, \quad t \in [a, \tau_1^*), \\ g(t, y_\infty^{[1]*}(t) &\neq 0, \quad t \in [\tau_1^*, \tau_2^*), \\ g(t, y_\infty^{[2]*}(t) &\neq 0, \quad t \in [\tau_2^*, b], \end{aligned} \tag{5.11}$$

where x_∞^* and $y_\infty^{[k]*}$, $k = 1, 2$, are functions (4.7), (4.17) with the parameter values (5.10). Then, due to (5.4), we can conclude that the vector function

$$u_\infty^*(t) := \begin{cases} x_\infty^*(t) & \text{if } t \in [a, \tau_1^*], \\ y_\infty^{[1]*}(t) & \text{if } t \in (\tau_1^*, \tau_2^*], \\ y_\infty^{[2]*}(t) & \text{if } t \in (\tau_2^*, b], \end{cases}$$

is a solution of the impulsive boundary value problem (1.1), (1.2), (1.3) with two jumps at the time instants τ_1^* and τ_2^*.

(d) If (5.11) does not hold, then values (5.10) do not correspond to a solution u of problem (1.1), (1.2), (1.3) satisfying $u(a) \in \Omega_0$ and having two jumps, and we should seek for other roots of (5.9). In the case of failure to find suitable roots, we proceed by looking for solutions with more jumps.

(3) We continue the computation for other values of $p \geq 3$.
The practical realization of this scheme is discussed in Sections 7 and 8 and illustrated in Section 10.

6 Dependence on the parameters

Under natural assumptions, the dependence of the iterations on the parameters is Lipschitzian. Indeed, assume that there exist non-negative matrices N_k, $k = 1, 2, \ldots, p$, such that

$$|v_1 - v_2 + \gamma_{\tau_k}(v_1) - \gamma_{\tau_k}(v_2)| \leq N_k |v_1 - v_2| \tag{6.1}$$

for all $\{v_1, v_2\} \subset \Omega_k$, $k = 1, 2, \ldots, p$. Given any continuous vector function

$$z : [\tau_k, \tau_{k+1}) \to \mathbb{R}^n, \quad k = 1, 2, \ldots, p,$$

put

$$(\mathcal{M}_k z)(t) := \beta_0(t; \tau_k, \tau_{k+1}) \int_{\tau_k}^{t} z(s)\,ds + (1 - \beta_0(t; \tau_k, \tau_{k+1})) \int_{t}^{\tau_{k+1}} z(s)\,ds,$$

where

$$\beta_0(t; \tau_k, \tau_{k+1}) := 1 - \frac{t - \tau_k}{\tau_{k+1} - \tau_k} \tag{6.2}$$

for $t \in [\tau_k, \tau_{k+1})$. Obviously, $\mathcal{M}_k$ is a positive linear operator with respect to the pointwise and componentwise partial ordering of the space of continuous functions. Based on (6.2), for all $j = 1, 2, \ldots$, put

$$\beta_j(t; \tau_k, \tau_{k+1}) := e_1^* \mathcal{M}_k(\beta_{j-1}(\cdot\,; \tau_k, \tau_{k+1}) e_1)(t), \quad t \in [\tau_k, \tau_{k+1}), \tag{6.3}$$

where e_1 stands for the first column of the unit matrix. It is easy to see that (6.3) defines the same recurrence relation as (2.6), the difference being the starting function only ((6.2) instead of the constant 1).

Lemma 6.1. *Let $m \geq 0$, $k \in \{1, 2, \ldots, p\}$, $\mu \in \Omega_{k+1}$ be fixed. Then*

$$\left|y_m^{[k]}(t; \tau_k, \tau_{k+1}, \bar{\lambda}, \mu) - y_m^{[k]}(t; \tau_k, \tau_{k+1}, \bar{\bar{\lambda}}, \mu)\right| \leq \sum_{i=0}^{m} \beta_i(t; \tau_k, \tau_{k+1}) K_k^i N_k |\bar{\lambda} - \bar{\bar{\lambda}}| \tag{6.4}$$

for all $t \in (\tau_k, \tau_{k+1})$ and $\{\bar{\lambda}, \bar{\bar{\lambda}}\} \subset \Omega_k$.

Proof. Let us fix $k \in \{1, 2, \ldots, p\}$, $\mu \in \Omega_{k+1}$, $t \in (\tau_k, \tau_{k+1})$, and $\{\bar{\lambda}, \bar{\bar{\lambda}}\} \subset \Omega_k$. Given any $m \geq 0$, we put

$$\mathbf{y}_m^{[k]}(t; \lambda) := y_m^{[k]}(t; \tau_k, \tau_{k+1}, \lambda, \mu), \quad \lambda \in \{\bar{\lambda}, \bar{\bar{\lambda}}\},$$

for the sake of brevity.

Considering formula (4.14) for $y_0^{[k]}$ and using notation (6.2), we find

$$y_0^{[k]}(t; \tau_k, \tau_{k+1}, \bar{\lambda}, \mu) - y_0^{[k]}(t; \tau_k, \tau_{k+1}, \bar{\bar{\lambda}}, \mu) = \beta_0(t; \tau_k, \tau_{k+1})(\bar{\lambda} - \bar{\bar{\lambda}} + \gamma_{\tau_k}(\bar{\lambda}) - \gamma_{\tau_k}(\bar{\bar{\lambda}})),$$

whence, by virtue of (6.1),

$$|\mathbf{y}_0^{[k]}(t, \bar{\lambda}) - \mathbf{y}_0^{[k]}(t, \bar{\bar{\lambda}})| \leq \beta_0(t; \tau_k, \tau_{k+1}) N_k |\bar{\lambda} - \bar{\bar{\lambda}}|, \tag{6.5}$$

which coincides with (6.4) for $m = 0$. It follows from (4.13) that, for any $m \geq 1$,

$$\begin{aligned}\mathbf{y}_m^{[k]}(t, \bar{\lambda}) - \mathbf{y}_m^{[k]}(t, \bar{\bar{\lambda}}) = \mathbf{y}_0^{[k]}(t, \bar{\lambda}) - \mathbf{y}_0^{[k]}(t, \bar{\bar{\lambda}}) + \beta_0(t; \tau_k, \tau_{k+1}) \int_{\tau_k}^{t} [f(s, \mathbf{y}_{m-1}^{[k]}(s, \bar{\lambda})) - f(s, \mathbf{y}_{m-1}^{[k]}(s, \bar{\bar{\lambda}}))]\, ds \\ - (1 - \beta_0(t; \tau_k, \tau_{k+1})) \int_{t}^{\tau_{k+1}} [f(s, \mathbf{y}_{m-1}^{[k]}(s, \bar{\lambda})) - f(s, \mathbf{y}_{m-1}^{[k]}(s, \bar{\bar{\lambda}}))]\, ds.\end{aligned}$$

Therefore, using (6.2), (6.5) and the Lipschitz condition (4.16), we obtain

$$|\mathbf{y}_m^{[k]}(t, \bar{\lambda}) - \mathbf{y}_m^{[k]}(t, \bar{\bar{\lambda}})| \leq \beta_0(t; \tau_k, \tau_{k+1}) N_k |\bar{\lambda} - \bar{\bar{\lambda}}| + K_k (\mathcal{M}_k[|\mathbf{y}_{m-1}^{[k]}(s, \bar{\lambda}) - \mathbf{y}_{m-1}^{[k]}(s, \bar{\bar{\lambda}})|])(t). \tag{6.6}$$

Assume that (6.4) holds for a certain $m = m_0$, $m_0 > 1$. Then, using (6.6), (6.7), and the identity

$$\mathcal{M}_k \beta_i(\cdot\,; \tau_k, \tau_{k+1})\, \lambda = \beta_{i+1}(\cdot\,; \tau_k, \tau_{k+1})\, \lambda, \tag{6.7}$$

which is an immediate consequence of (6.2) and (6.3) for any constant vector λ and integer $i \geq 0$, we get

$$\begin{aligned}|\mathbf{y}_{m_0+1}^{[k]}(t, \bar{\lambda}) - \mathbf{y}_{m_0+1}^{[k]}(t, \bar{\bar{\lambda}})| &\leq \beta_0(t; \tau_k, \tau_{k+1}) N_k |\bar{\lambda} - \bar{\bar{\lambda}}| + K_k \mathcal{M}_k [|\mathbf{y}_{m_0}^{[k]}(\cdot\,, \bar{\lambda}) - \mathbf{y}_{m_0}^{[k]}(\cdot\,, \bar{\bar{\lambda}})|](t) \\ &\leq \beta_0(t; \tau_k, \tau_{k+1}) N_k |\bar{\lambda} - \bar{\bar{\lambda}}| + K_k N_k \mathcal{M}_k ([\beta_0(\cdot\,; \tau_k, \tau_{k+1}) 1_n \\ &\qquad + \cdots + \beta_{m_0}(\cdot\,; \tau_k, \tau_{k+1}) K_k^{m_0}] N_k |\bar{\lambda} - \bar{\bar{\lambda}}|)(t) \\ &= \sum_{i=0}^{m_0+1} \beta_i(t; \tau_k, \tau_{k+1}) K_k^i N_k |\bar{\lambda} - \bar{\bar{\lambda}}|\end{aligned}$$

for all $t \in [\tau_k, \tau_{k+1})$. The last relation, in view of the arbitrariness of m_0, proves the lemma. □

Arguing by analogy and using (4.2) and (4.3), one proves similar estimates with respect to other function arguments. For example,

Lemma 6.2. *Let $m \geq 0$, $k \in \{1, 2, \ldots, p\}$, $\xi \in \Omega_0$ be fixed. Then*

$$\left|x_m(t; \tau_1, \xi, \bar{\mu}) - x_m(t; \tau_1, \xi, \bar{\bar{\mu}})\right| \leq \sum_{i=0}^{m} \beta_i(t; a, \tau_1) K_k^i |\bar{\mu} - \bar{\bar{\mu}}|$$

for all $t \in (a, \tau_1)$, $\{\bar{\mu}, \bar{\bar{\mu}}\} \subset \Omega_1$.

Note that $\sum_{i=0}^{m} \beta_i(t; \tau_k, \tau_{k+1}) K_k^i \leq L_{m,k}$ for $t \in (\tau_k, \tau_{k+1})$, where

$$L_{m,k} = 1_n + \frac{5}{9} \sum_{i=1}^{m} \left(\frac{3}{10}\right)^i (\tau_{k+1} - \tau_k)^{i+1} K_k^i.$$

7 Approximation of solutions

Let us fix $m \in \mathbb{N}$. The solvability of the determining system (5.1) can be examined by using its approximate version

$$
\begin{aligned}
\Psi_{a,m} &:= \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, x_m(s))\,ds = 0,\\
\Psi_{k,m} &:= \lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y_m^{[k]}(s))\,ds = 0, \quad k = 1, 2, \dots, p,\\
& g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, \dots, p,\\
& V(\xi, \lambda^{[p+1]}) = 0,
\end{aligned}
\tag{7.1}
$$

with the additional conditions

$$
\begin{aligned}
g(t, x_m(t)) &\neq 0, \quad t \in [a, \tau_1),\\
g(t, y_m^{[k]}(t)) &\neq 0, \quad t \in [\tau_k, \tau_{k+1}),\ k = 1, 2, \dots, p-1,\\
g(t, y_m^{[p]}(t)) &\neq 0, \quad t \in [\tau_p, b],
\end{aligned}
\tag{7.2}
$$

where

$$a < \tau_1 < \tau_2 < \cdots < \tau_{p-1} < \tau_p < \tau_{p+1} = b.$$

Clearly, the approximate determining system (7.1) is obtained from the exact system (5.1) by replacing the limits x_∞ and $y_\infty^{[k]}$ from (4.7) and (4.17) by the iterations x_m and $y_m^{[k]}$ from (4.3) and (4.13), respectively. It is important that, in contrast to (5.1), all the terms involved in (7.1) and (7.2) can be constructed explicitly.

Assume that the values

$$\hat\tau_1, \hat\tau_2, \dots, \hat\tau_p \in (a, b), \qquad \hat\xi \in \Omega_0, \qquad \hat\lambda^{[k]} \in \Omega_k, \quad k = 1, \dots, p+1, \tag{7.3}$$

are a solution of system (7.1). Consider the functions

$$\hat x_m(\cdot) := x_m(\cdot\,; \hat\tau_1, \hat\xi, \hat\lambda^{[1]})$$

and

$$\hat y_m^{[k]}(\cdot) := y_m^{[k]}(\cdot\,; \hat\tau_k, \hat\tau_{k+1}, \hat\lambda^{[k]}, \hat\lambda^{[k+1]}),$$

$k = 1, 2, \dots, p$, determined according to (4.3) and (4.13) by values (7.3). If $\hat x_m$ and $\hat y_m^{[k]}$ satisfy (7.2) with $\tau_k = \hat\tau_k$, $k = 1, \dots, p$, then the vector function

$$
\hat u(t) := \begin{cases} \hat x_m(t), & t \in [a, \hat\tau_1],\\ \hat y_m^{[k]}(t), & t \in (\hat\tau_k, \hat\tau_{k+1}],\ k = 1, 2, \dots, p-1,\\ \hat y_m^{[p]}(t), & t \in (\hat\tau_p, b], \end{cases}
\tag{7.4}
$$

undergoes a jump of the value $\gamma_{\tau_k}(\hat\lambda^{[k]})$ at the moment $\hat\tau_k$, $k = 1, \dots, p$. In view of Theorems 4.2 and 4.4, the following estimates hold for $\hat x_m$ and $\hat y_m^{[k]}$:

$$\left|x_\infty(t; \hat\tau_1, \hat\xi, \hat\lambda^{[1]}) - \hat x_m(t)\right| \le \frac{5}{9}\alpha_1(t; a, \hat\tau_1 - a) Q_0^m (1_n - Q_0)^{-1} \delta_f(\mathcal{U}_0) \tag{7.5}$$

for $t \in [a, \hat\tau_1]$ and

$$\left|y_\infty^{[k]}(t; \hat\tau_k, \hat\tau_{k+1}, \hat\lambda^{[k]}, \hat\lambda^{[k+1]}) - \hat y_m^{[k]}(t)\right| \le \frac{5}{9}\alpha_1(t; \hat\tau_k, \hat\tau_{k+1} - \hat\tau_k) Q_k^m (1_n - Q_k)^{-1} \delta_f(\mathcal{U}_k) \tag{7.6}$$

for $t \in (\hat\tau_k, \hat\tau_{k+1}]$, $k = 1, \dots, p$.

By construction, $\hat\xi$ and $\hat\lambda^{[i]}$, $i = 1, \dots, p$, coincide with the values of iterations at the corresponding nodes. Therefore, assuming the solvability of the approximate determining equations, one can claim that the convergence of iterations implies convergence of the roots of mth determining equations as m tends to ∞.

Thus, $\hat{\tau}_1 \approx \tau_1^*$, $\hat{\xi} \approx \xi^*$ and $\hat{\lambda}^{[k]} \approx \lambda^{[k]*}$, $k = 1, 2, \dots, p$, where the values with a star stand for the roots of the exact determining equations, and the accuracy of approximation increases with the number of iteration, m. On the other hand, it follows from the estimates of Section 6 that the dependence of x_∞ and $y_\infty^{[k]}$ on parameters (3.7) is Lipschitzian and, in particular, continuous. Therefore,

$$x_m(\cdot\,; \hat{\tau}_1, \hat{\xi}, \hat{\lambda}^{[1]}) \approx x_m(\cdot\,; \tau_1^*, \xi^*, \lambda^{[1]*}), \tag{7.7}$$

$$y_m^{[k]}(\cdot\,; \hat{\tau}_k, \hat{\tau}_{k+1}, \hat{\lambda}^{[k]}, \hat{\lambda}^{[k+1]}) \approx y_m^{[k]}(\cdot\,; \tau_k^*, \tau_{k+1}^*, \lambda^{[k]*}, \lambda^{[k+1]*}) \tag{7.8}$$

for $k = 1, 2, \dots, p$, and the accuracy of approximation grows with m (i.e., the corresponding differences tend to 0). Note that, in (7.7) and (7.8), the terms on the right represent an exact solution of the problem. The obvious inequalities

$$\begin{aligned}\left|x_\infty(t; \tau_1^*, \xi^*, \lambda^{[1]*}) - \hat{x}_m(t)\right| \le{}& \left|x_\infty(t; \tau_1^*, \xi^*, \lambda^{[1]*}) - x_m(t; \tau_1^*, \xi^*, \lambda^{[1]*})\right| \\ &+ \left|x_m(t; \tau_1^*, \xi^*, \lambda^{[1]*}) - \hat{x}_m(t)\right|, \quad t \in [a, \hat{\tau}_1],\end{aligned}$$

and

$$\begin{aligned}&\left|y_\infty^{[k]}(t; \tau_k^*, \tau_{k+1}^*, \lambda^{[k]*}, \lambda^{[k+1]*}) - \hat{y}_m^{[k]}(t)\right| \\ &\quad \le \left|y_\infty^{[k]}(t; \tau_k^*, \tau_{k+1}^*, \lambda^{[k]*}, \lambda^{[k+1]*}) - y_m^{[k]}(t; \tau_k^*, \tau_{k+1}^*, \lambda^{[k]*}, \lambda^{[k+1]*})\right| \\ &\quad\quad + \left|y_m^{[k]}(t; \tau_k^*, \tau_{k+1}^*, \lambda^{[k]*}, \lambda^{[k+1]*}) - \hat{y}_m^{[k]}(t)\right|, \quad t \in (\hat{\tau}_k, \hat{\tau}_{k+1}],\end{aligned}$$

ensure, in view of (7.5)–(7.8), that $\hat{x}_m(\cdot) \to x_\infty(\cdot\,; \tau_1^*, \xi^*, \lambda^{[1]*})$ and $\hat{y}_m^{[k]}(\cdot) \to y_\infty^{[k]}(\cdot\,; \tau_k^*, \tau_{k+1}^*, \lambda^{[k]*}, \lambda^{[k+1]*})$ as $m \to \infty$. This allows one to regard (7.4) as the mth approximation to a solution of problem (1.1), (1.2), (1.3) with p jumps. The solvability analysis based on the properties of the approximate determining system (7.1) can be carried out using the topological degree methods as it is done in [8, 10] for problems without impulses. This topic is not treated here.

8 Frozen parameter scheme

The simplest way how to choose the parameter sets (3.2) is to take a compact convex set $\Omega_0 \subset \mathbb{R}^n$ and put

$$\Omega_0 = \Omega_1,\ \Omega_1^+ = \Omega_2,\ \dots,\ \Omega_{p-1}^+ = \Omega_p,\ \Omega_p^+ = \Omega_{p+1}, \tag{8.1}$$

where

$$\Omega_k^+ = \{x + \gamma_{\tau_k}(x) : x \in \Omega_k\}, \quad k = 1, \dots, p. \tag{8.2}$$

Then the sets appearing in (3.3) and (3.4) have the form

$$\begin{aligned}\mathcal{U}_0 &= \bigcup_{v \in \Omega_0} B(v, \varrho^{[0]}), \\ \mathcal{U}_k &= \bigcup_{v \in \Omega_k^+} B(v, \varrho^{[k]}), \quad k = 1, \dots, p.\end{aligned} \tag{8.3}$$

If the assumptions of Theorems 4.2 and 4.4 are satisfied on sets (8.1), (8.2), (8.3), we can suggest the following algorithm for approximate solution of problem (1.1), (1.2), (1.3) with p jumps using *frozen parameters*.

(1) Using (4.2), introduce a vector function x_0 depending on the parameters τ_1, ξ, and $\lambda^{[1]}$ of form (3.7). Then compute the first iteration x_1 according to formula (4.3). Similarly, for $k = 1, \dots, p$, use formula (4.12) and introduce a vector function $y_0^{[k]}$ depending on the parameters $\tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}$ from (3.7). Compute the corresponding first iteration $y_1^{[k]}$ according to (4.13).
(2) Put $m = 1$ in the approximate determining system (7.1), find its solution called *first frozen parameters* and write it down as in (7.3).

(3) Using x_1 and $y_1^{[k]}$, $k = 1, \dots, p$, constructed in Step (1) and the first frozen parameters found on Step (2), introduce the vector functions

$$\begin{aligned} X_1(t) &:= x_1(t; \hat{\tau}_1, \hat{\xi}, \hat{\lambda}^{[1]}), && t \in [a, \tau_1], \\ Y_1^{[k]}(t) &:= y_1^{[k]}(t; \hat{\tau}_k, \hat{\tau}_{k+1}, \hat{\lambda}^{[k]}, \hat{\lambda}^{[k+1]}), && t \in [\tau_k, \tau_{k+1}],\ k = 1, \dots, p-1, \\ Y_1^{[p]}(t) &:= y_1^{[p]}(t; \hat{\tau}_p, b, \hat{\lambda}^{[p]}, \hat{\lambda}^{[p+1]}), && t \in [\tau_p, b]. \end{aligned}$$

(4) Define the *second frozen iterations*

$$\begin{aligned} \hat{x}_2(t) &:= \hat{x}_2(t; \tau_1, \xi, \lambda^{[1]}), && t \in [a, \tau_1], \\ \hat{y}_2^{[k]}(t) &:= \hat{y}_2^{[k]}(t; \tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}), && t \in [\tau_k, \tau_{k+1}],\ k = 1, \dots, p, \end{aligned}$$

according to (4.2), (4.3) and (4.12), (4.13) as follows:

$$\hat{x}_2(t) = x_0(t) + \int_a^t f(s, X_1(s))\,ds - \frac{t-a}{\tau_1 - a}\int_a^{\tau_1} f(s, X_1(s))\,ds$$

for $t \in [a, \tau_1]$ and

$$\hat{y}_2^{[k]}(t) = y_0^{[k]}(t) + \int_{\tau_k}^t f(s, Y_1^{[k]}(s))\,ds - \frac{t-\tau_k}{\tau_{k+1} - \tau_k}\int_{\tau_k}^{\tau_{k+1}} f(s, Y_1^{[k]}(s))\,ds$$

for $t \in [\tau_k, \tau_{k+1}]$, $k = 1, \dots, p$. Thus, the first iterations x_1 and $y_1^{[k]}$ in (4.3) and (4.13) are replaced here, respectively, by X_1 and $Y_1^{[k]}$ introduced in Step (3).

(5) Put $m = 2$, modify the approximate system of determining equations (7.1) by substituting there the second frozen iterations $\hat{x}_2$ and $\hat{y}_2^{[k]}$ from Step (4). The resulting modified system of $(p + 2)n + p$ scalar algebraic equations has the form

$$\begin{aligned} \hat{\Psi}_{a,2} &:= \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, \hat{x}_2(s))\,ds = 0, \\ \hat{\Psi}_{k,2} &:= \lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, \hat{y}_2^{[k]}(s))\,ds = 0, \quad k = 1, 2, \dots, p, \\ & g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, \dots, p, \\ & V(\xi, \lambda^{[p+1]}) = 0. \end{aligned} \tag{8.4}$$

Find a solution of (8.4) called the *second frozen parameters* and write it down as in (7.3).

(6) Using $\hat{x}_2$ and $\hat{y}_2^{[k]}$, $k = 1, \dots, p$, constructed in Step (4) and the second frozen parameters found in Step (5), introduce the vector functions

$$\begin{aligned} X_2(t) &:= \hat{x}_2(t; \hat{\tau}_1, \hat{\xi}, \hat{\lambda}^{[1]}), && t \in [a, \tau_1], \\ Y_2^{[k]}(t) &:= \hat{y}_2^{[k]}(t; \hat{\tau}_k, \hat{\tau}_{k+1}, \hat{\lambda}^{[k]}, \hat{\lambda}^{[k+1]}), && t \in [\tau_k, \tau_{k+1}],\ k = 1, \dots, p-1, \\ Y_2^{[p]}(t) &:= \hat{y}_2^{[p]}(t; \hat{\tau}_p, b, \hat{\lambda}^{[p]}, \hat{\lambda}^{[p+1]}), && t \in [\tau_p, b]. \end{aligned}$$

(7) Define the *third frozen iterations*

$$\begin{aligned} \hat{x}_3(t) &:= \hat{x}_3(t; \tau_1, \xi, \lambda^{[1]}), && t \in [a, \tau_1], \\ \hat{y}_3^{[k]}(t) &:= \hat{y}_3^{[k]}(t; \tau_k, \tau_{k+1}, \lambda^{[k]}, \lambda^{[k+1]}), && t \in [\tau_k, \tau_{k+1}],\ k = 1, \dots, p, \end{aligned}$$

by putting

$$\hat{x}_3(t) = x_0(t) + \int_a^t f(s, X_2(s))\,ds - \frac{t-a}{\tau_1 - a}\int_a^{\tau_1} f(s, X_2(s))\,ds$$

for $t \in [a, \tau_1]$ and

$$\hat{y}_3^{[k]}(t) = y_0^{[k]}(t) + \int_{\tau_k}^{t} f(s, Y_2^{[k]}(s))\,\mathrm{d}s - \frac{t-\tau_k}{\tau_{k+1}-\tau_k}\int_{\tau_k}^{t_{k+1}} f(s, Y_2^{[k]}(s))\,\mathrm{d}s$$

for $t \in [\tau_k+, \tau_{k+1}]$, $k = 1, \dots, p$. Hence, by analogy, the second iterations x_2 and $y_2^{[k]}$ in (4.3) and (4.13) are replaced by X_2 and $Y_2^{[k]}$, respectively.

(8) Put $m = 3$, modify system (7.1) by substituting there the third frozen iterations $\hat{x}_3$ and $\hat{y}_3^{[k]}$ from Step (7). The resulting modified system of $(p+2)n + p$ scalar algebraic equations has the form

$$\begin{aligned}\hat{\Psi}_{a,3} &:= \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, \hat{x}_3(s))\,\mathrm{d}s = 0,\\ \hat{\Psi}_{k,3} &:= \lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, \hat{y}_3^{[k]}(s))\,\mathrm{d}s = 0, \quad k = 1, 2, \dots, p,\\ g(\tau_k, \lambda^{[k]}) &= 0, \quad k = 1, \dots, p,\\ V(\xi, \lambda^{[p+1]}) &= 0.\end{aligned} \tag{8.5}$$

Find a solution of system (8.5) called the *third frozen parameters* and write it down as in (7.3).

(9) Continue in a similar manner and derive higher frozen parameters and higher frozen iterations. If, for some $m > 2$, the mth and $(m-1)$th frozen parameters are close enough to one another, we put

$$\begin{aligned} X_m(t) &:= \hat{x}_m(t; \hat{\tau}_1, \hat{\xi}, \hat{\lambda}^{[1]}), && t \in [a, \hat{\tau}_1],\\ Y_m^{[k]}(t) &:= \hat{y}_m^{[k]}(t; \hat{\tau}_k, \hat{\tau}_{k+1}, \hat{\lambda}^{[k]}, \hat{\lambda}^{[k+1]}), && t \in [\hat{\tau}_k, \hat{\tau}_{k+1}],\ k = 1, \dots, p-1,\\ Y_m^{[p]}(t) &:= \hat{y}_m^{[p]}(t; \hat{\tau}_p, b, \hat{\lambda}^{[p]}, \hat{\lambda}^{[p+1]}), && t \in [\hat{\tau}_p, b],\end{aligned}$$

and, according to (1.6), verify the conditions

$$\begin{aligned} g(t, X_m(t)) &\neq 0, \quad t \in [a, \hat{\tau}_1),\\ g(t, Y_m^{[k]}(t)) &\neq 0, \quad t \in [\hat{\tau}_k, \hat{\tau}_{k+1}),\ k = 1, 2, \dots, p-1,\\ g(t, Y_m^{[p]}(t)) &\neq 0, \quad t \in [\hat{\tau}_p, b].\end{aligned} \tag{8.6}$$

If (8.6) is fulfilled, then the function

$$\hat{u}(t) := \begin{cases} X_m(t), & t \in [a, \hat{\tau}_1],\\ Y_m^{[k]}(t), & t \in (\hat{\tau}_k, \hat{\tau}_{k+1}],\ k = 1, 2, \dots, p-1,\\ Y_m^{[p]}(t), & t \in (\hat{\tau}_p, b],\end{cases}$$

is regarded as the mth approximation of a solution u of problem (1.1), (1.2), (1.3) with $u(a) \in \Omega_0$ and p jumps. If (8.6) is not satisfied, then we consider the frozen parameter scheme with other numbers of jumps as in Section 5.1.

Another possible algorithm, which could be adopted for practical computations of approximate solutions for problem (1.1), (1.2), (1.3), is the scheme with a polynomial interpolation presented in [10] for a Dirichlet problem without impulses.

9 Zeroth approximation

We see that the sets in (8.1)–(8.3) are determined by the set Ω_0, which contains all possible starting points of solutions of problem (1.1), (1.2), (1.3) with p jumps, and by the vectors $\varrho^{[0]}$ and $\varrho^{[k]}$, $k = 1, \dots, p$. The choice of Ω_0 in concrete cases may be motivated by the nature of a practical problem modelled by (1.1), (1.2), (1.3), due to which one may expect that some values should stay in a certain range. On the other hand, assumptions imposed on the set Ω_0 are just those of Theorems 4.2, 4.4 and 5.1, and we can try to choose among many possibilities in every concrete situation.

In any case, it is convenient to start our computation directly at $m = 0$, where no iterations are present and one thus works only with the functions x_0 and $y_0^{[k]}$, $k = 1, \dots, p$, given by (4.2) and (4.12). Being piecewise linear functions (in fact, it is clear from (4.2) and (4.12) that they are just broken lines joining the points (a, ξ), $(\tau_k, \lambda^{[k]})$, $k = 1, \dots, p$, and $(b, \lambda^{[p+1]})$), these zeroth approximations are very rough but, nevertheless, they are usually rather helpful when choosing the sets where it is natural to look for the parameter values.

Computing the roots $\hat{\xi}, \hat{\tau}_1, \dots, \hat{\tau}_p, \hat{\lambda}^{[1]}, \dots, \hat{\lambda}^{[p+1]}$ of the zeroth approximate determining system, which consists of $(p+2)n + p$ scalar algebraic equations and has the form

$$\begin{aligned} \Psi_{a,0} &:= \lambda^{[1]} - \xi - \int_a^{\tau_1} f(s, x_0(s))\,\mathrm{d}s = 0, \\ \Psi_{k,0} &:= \lambda^{[k+1]} - \lambda^{[k]} - \gamma_{\tau_k}(\lambda^{[k]}) - \int_{\tau_k}^{\tau_{k+1}} f(s, y_0^{[k]}(s))\,\mathrm{d}s = 0, \quad k = 1, 2, \dots, p, \\ & g(\tau_k, \lambda^{[k]}) = 0, \quad k = 1, \dots, p, \\ & V(\xi, \lambda^{[p+1]}) = 0, \end{aligned} \tag{9.1}$$

we get a hint that makes it much easier to choose the set Ω_0 as a neighbourhood of $\hat{\xi}$ in a reasonable way and thus avoid unnecessary computations on sets that might possibly be excessively large. A quick glance at the graph of the zeroth approximation

$$\hat{u}_0(t) := \begin{cases} \hat{x}_0(t), & t \in [a, \hat{\tau}_1], \\ \hat{y}_0^{[k]}(t), & t \in (\hat{\tau}_k, \hat{\tau}_{k+1}],\ k = 1, 2, \dots, p-1, \\ \hat{y}_0^{[p]}(t), & t \in (\hat{\tau}_p, b], \end{cases}$$

which is obtained in a straightforward way with a minimal computational effort, helps us to understand where the graph of the solution in question may possibly be located.

10 Examples

Let us apply the numeric-analytic techniques described above for some examples. Put $n = 2$ and consider the system of two differential equations

$$\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = u_2^2(t) - u_1^2(t) + t, \qquad \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = u_1^2(t) - u_2^2(t) - t \tag{10.1}$$

on the interval $[0, 0.5]$ under the non-linear two-point boundary conditions

$$u_1^2(0) + u_2(0.5) + 0.125 = 0, \qquad u_1^2(0.5) + u_2(0) - 0.015625 = 0. \tag{10.2}$$

10.1 Barrier (10.3) and two jumps

Put $p = 2$, consider the barrier

$$G = \{(t, x) \in [0, 0.5] \times \mathbb{R}^2 : x_1 - 7.233\bar{3}t^2 + 2.368\bar{3}t - 0.04 = 0\} \tag{10.3}$$

and the state-dependent impulse conditions at two unknown points τ_1 and τ_2

$$\begin{aligned} u_1(\tau_1+) - u_1(\tau_1) &= 0.01, \qquad u_2(\tau_1+) - u_2(\tau_1) = -0.01, \\ u_1(\tau_2+) - u_1(\tau_2) &= 0.015, \quad\ u_2(\tau_2+) - u_2(\tau_2) = -0.015, \end{aligned} \tag{10.4}$$

where, by (1.6), τ_1 and τ_2 have to satisfy the conditions

$$\begin{aligned} u_1(\tau_k) - 7.233\bar{3}\tau_k^2 + 2.368\bar{3}\tau_k - 0.04 &= 0, \quad k = 1, 2, \\ u_1(t) - 7.233\bar{3}t^2 + 2.368\bar{3}t - 0.04 &\neq 0, \quad t \in [0, 0.5] \setminus \{\tau_1, \tau_2\}. \end{aligned} \tag{10.5}$$

We are interested in a solution of problem (10.1), (10.4), (10.5), (10.2) as defined in Definition 1.1 with $n = p = 2$. Here, $a = 0$, $b = 0.5$ and $f = \mathrm{col}(f_1, f_2)$, where

$$f_1(t, x_1, x_2) = x_2^2 - x_1^2 + t, \quad f_2(t, x_1, x_2) = x_1^2 - x_2^2 - t$$

for all (t, x_1, x_2). The impulse vector functions γ_{τ_1} and γ_{τ_2} in (1.7) are constant here, namely,

$$\gamma_{\tau_1} = \mathrm{col}(0.01, -0.01), \quad \gamma_{\tau_2} = \mathrm{col}(0.015, -0.015), \tag{10.6}$$

the barrier function g is given by the equality

$$g(t, x) = x_1 - 7.233\bar{3}t^2 + 2.368\bar{3}t - 0.04,$$

and the components of the vector function $V = \mathrm{col}(V_1, V_2)$ determining the boundary conditions have the form

$$\begin{aligned} V_1(x_1, x_2, y_1, y_2) &= x_1^2 + y_2 + 0.125, \\ V_2(x_1, x_2, y_1, y_2) &= y_1^2 + x_2 - 0.015625. \end{aligned}$$

Let us describe in detail the steps of our method. Recalling the remarks in Section 9, we start by introducing the zeroth iterations x_0, $y_0^{[k]}$, $k = 1, 2$, and solving the corresponding zeroth approximate determining system (9.1) consisting of ten algebraic equations. A numerical computation gives us the roots $\hat{\tau}_1$, $\hat{\tau}_2$, $\hat{\xi}_1$, $\hat{\xi}_2$, $\hat{\lambda}_1^{[k]}$, $\hat{\lambda}_2^{[k]}$, $k = 1, \dots, 3$, presented in the first column of Table 1. Since, in this case, we have

$$\hat{\xi}_1 = -0.1059217222, \quad \hat{\xi}_2 = 0.01369007648,$$

it is natural to take, e.g.,

$$\Omega_0 = [-0.14, 0.04] \times [-0.18, 0.03] = \Omega_1,$$

and then, by (8.1), (8.2), (10.6), we obtain

$$\begin{aligned} \Omega_1^+ &= [-0.15, 0.05] \times [-0.19, 0.04] = \Omega_2, \\ \Omega_2^+ &= [-0.165, 0.065] \times [-0.205, 0.055] = \Omega_3. \end{aligned}$$

Now choose the vectors

$$\begin{aligned} \varrho^{[0]} &= \mathrm{col}(0.1, 0.1), \\ \varrho^{[1]} &= \mathrm{col}(0.15, 0.15), \\ \varrho^{[2]} &= \mathrm{col}(0.1, 0.15), \end{aligned}$$

and, using (8.3), construct the sets

$$\begin{aligned} \mathcal{U}_0 &= [-0.24, 0.14] \times [-0.28, 0.13], \\ \mathcal{U}_1 &= [-0.30, 0.20] \times [-0.34, 0.19], \\ \mathcal{U}_2 &= [-0.265, 0.165] \times [-0.355, 0.205]. \end{aligned}$$

Carrying out computations in Maple, we find that conditions (4.4) and (4.5) are fulfilled with the matrix

$$K_0 = \begin{pmatrix} 0.56 & 0.48 \\ 0.48 & 0.56 \end{pmatrix}$$

and conditions (4.15) and (4.16), $k = 1, 2$, are fulfilled with the matrices

$$K_1 = \begin{pmatrix} 0.68 & 0.6 \\ 0.6 & 0.68 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0.1065 & 0.0795 \\ 0.0795 & 0.1065 \end{pmatrix}.$$

Therefore, the frozen parameter scheme described in Section 8 can be applied using both symbolic and numerical computations in MAPLE. We thus proceed as follows.

(1) Compute the first iterations x_1, $y_1^{[1]}$ and $y_1^{[2]}$.

(2) Put $m = 1$ and solve system (7.1) which consists of ten scalar algebraic equations with the unknowns τ_1, τ_2, ξ_1, ξ_2, and $\lambda_1^{[1]}$, $\lambda_2^{[1]}$, $\lambda_1^{[2]}$, $\lambda_2^{[2]}$, $\lambda_1^{[3]}$, and $\lambda_2^{[3]}$. Under the restrictions $\tau_1 \in (0.00001, 0.25)$ and $\tau_2 \in (0.25, 0.5)$, numerical computations in MAPLE give the roots (called in the sequel "first frozen parameters" for the sake of brevity) shown in the second column of Table 1.

(3) Introduce the vector functions $X_1 = \operatorname{col}(X_{11}, X_{12})$, $Y_1^{[1]} = \operatorname{col}(Y_{11}^{[1]}, Y_{12}^{[1]})$, and $Y_1^{[2]} = \operatorname{col}(Y_{11}^{[2]}, Y_{12}^{[2]})$ as follows. Using the first frozen parameters, put

$$\begin{aligned} X_1(t) &:= x_1(t; \hat\tau_1, \hat\xi, \hat\lambda^{[1]}), && t \in [0, \hat\tau_1],\\ Y_1^{[1]}(t) &:= y_1^{[1]}(t; \hat\tau_1, \hat\tau_2, \hat\lambda^{[1]}, \hat\lambda^{[2]}), && t \in [\hat\tau_1, \hat\tau_2],\\ Y_1^{[2]}(t) &:= y_1^{[2]}(t; \hat\tau_2, 0.5, \hat\lambda^{[2]}, \hat\lambda^{[3]}), && t \in [\hat\tau_2, 0.5], \end{aligned}$$

and get their componentwise form:

$$\begin{aligned} X_{11}(t) &= -0.1049467336 - 1.356666667 \cdot 10^{-11} t^3 + 0.5026541335 t^2 \\ &\qquad - 0.01092495168 t, && t \in [0, \hat\tau_1],\\ X_{12}(t) &= 0.01362740290 + 1.356666667 \cdot 10^{-11} t^3 - 0.5026541335 t^2 \\ &\qquad + 0.01092495181 t, && t \in [0, \hat\tau_1],\\ Y_{11}^{[1]}(t) &= -0.09496693961 + 0.5158065535 t^2 - 0.01171732438 t, && t \in [\hat\tau_1, \hat\tau_2],\\ Y_{12}^{[1]}(t) &= 0.00364760892 - 0.5158065535 t^2 + 0.01171732445 t, && t \in [\hat\tau_1, \hat\tau_2],\\ Y_{11}^{[2]}(t) &= -0.07909248307 + 0.5362580815 t^2 - 0.02055510225 t, && t \in [\hat\tau_2, 0.5],\\ Y_{12}^{[2]}(t) &= -0.0122268475 - 0.5362580815 t^2 + 0.02055510169 t, && t \in [\hat\tau_2, 0.5]. \end{aligned}$$

(4) Define the second frozen iterations $\hat x_2$, $\hat y_2^{[1]}$, and $\hat y_2^{[2]}$.

(5) Put $m = 2$, solve system (8.4) of ten scalar algebraic equations and compute the second frozen parameters. The values obtained are given in the first column of Table 2.

(6) Using the second frozen parameters, put

$$\begin{aligned} X_2(t) &:= \hat x_2(t; \hat\tau_1, \hat\xi, \hat\lambda^{[1]}), && t \in [0, \hat\tau_1],\\ Y_2^{[1]}(t) &:= \hat y_2^{[1]}(t; \hat\tau_1, \hat\tau_2, \hat\lambda^{[1]}, \hat\lambda^{[2]}), && t \in [\hat\tau_1, \hat\tau_2],\\ Y_2^{[2]}(t) &:= \hat y_2^{[2]}(t; \hat\tau_2, 0.5, \hat\lambda^{[2]}, \hat\lambda^{[3]}), && t \in [\hat\tau_2, 0.5]. \end{aligned}$$

(7) Define the third frozen iterations $\hat x_3$, $\hat y_3^{[1]}$, and $\hat y_3^{[2]}$.

(8) Put $m = 3$, solve system (8.5) of ten scalar algebraic equations and get the third frozen parameters. The values obtained are shown in the last column of Table 2.

(9) Using the third frozen parameters, put

$$\begin{aligned} X_3(t) &:= \hat x_3(t; \hat\tau_1, \hat\xi, \hat\lambda^{[1]}), && t \in [0, \hat\tau_1],\\ Y_3^{[1]}(t) &:= \hat y_3^{[1]}(t; \hat\tau_1, \hat\tau_2, \hat\lambda^{[1]}, \hat\lambda^{[2]}), && t \in [\hat\tau_1, \hat\tau_2],\\ Y_3^{[2]}(t) &:= \hat y_3^{[2]}(t; \hat\tau_2, 0.5, \hat\lambda^{[2]}, \hat\lambda^{[3]}), && t \in [\hat\tau_2, 0.5], \end{aligned}$$

and show that condition (8.6) with $p = 2$ holds for $m = 3$. More precisely, for $\hat\tau_1 = 0.07955623539$ and $\hat\tau_2 = 0.2787337381$,

$$\begin{aligned} X_{31}(t) - 7.233\bar3 t^2 + 2.368\bar3 t - 0.04 &\neq 0, && t \in [0, \hat\tau_1),\\ Y_{31}^{[1]}(t) - 7.233\bar3 t^2 + 2.368\bar3 t - 0.04 &\neq 0, && t \in [\hat\tau_1, \hat\tau_2),\\ Y_{31}^{[2]}(t) - 7.233\bar3 t^2 + 2.368\bar3 t - 0.04 &\neq 0, && t \in [\hat\tau_2, 0.5]. \end{aligned}$$

	$m = 0$	$m = 1$
$\hat{\tau}_1$	0.08032359386	0.07955621663
$\hat{\tau}_2$	0.278089169	0.2787337541
$\hat{\xi}_1$	−0.1059217222	−0.1049467336
$\hat{\xi}_2$	0.01369007648	0.0136274029
$\hat{\lambda}_1^{[1]}$	−0.1035644481	−0.1026344871
$\hat{\lambda}_2^{[1]}$	0.01133280237	0.01131515641
$\hat{\lambda}_1^{[2]}$	−0.05922824314	−0.05815864985
$\hat{\lambda}_2^{[2]}$	−0.03300340255	−0.03316068084
$\hat{\lambda}_1^{[3]}$	0.04398776554	0.0446944862
$\hat{\lambda}_2^{[3]}$	−0.1362194112	−0.1360138169

Table 1. Frozen parameters for problem (10.1), (10.2), (10.4), (10.5): $m = 0, 1$.

	$m = 2$	$m = 3$
$\hat{\tau}_1$	0.07955621664	0.07955623539
$\hat{\tau}_2$	0.2787337541	0.2787337381
$\hat{\xi}_1$	−0.1049467336	−0.1049467573
$\hat{\xi}_2$	0.0136274029	0.01362740444
$\hat{\lambda}_1^{[1]}$	−0.1026344871	−0.1026345099
$\hat{\lambda}_2^{[1]}$	0.01131515641	0.01131515702
$\hat{\lambda}_1^{[2]}$	−0.05815864985	−0.05815867642
$\hat{\lambda}_2^{[2]}$	−0.03316068084	−0.03316067649
$\hat{\lambda}_1^{[3]}$	0.0446944862	0.04469446897
$\hat{\lambda}_2^{[3]}$	−0.1360138169	−0.1360138219

Table 2. Frozen parameters for problem (10.1), (10.2), (10.4), (10.5): $m = 2, 3$.

Consequently, the vector function

$$\hat{u}_3(t) = \begin{cases} X_3(t) & \text{if } t \in [0, \hat{\tau}_1], \\ Y_3^{[1]}(t) & \text{if } t \in (\hat{\tau}_1, \hat{\tau}_2], \\ Y_3^{[2]}(t) & \text{if } t \in (\hat{\tau}_2, 0.5], \end{cases} \tag{10.7}$$

is the third approximation to a solution of problem (10.1), (10.2), (10.4), (10.5).

The graph of the third approximation $\hat{u}_3$ of a solution of problem (10.1), (10.2), (10.4), (10.5) and its orthogonal projection onto the (t, u_1) plane are presented in Figure 1. Figure 2 shows the graph of barrier (10.3) and the points where it is intersected by the graph of $\hat{u}_3$.

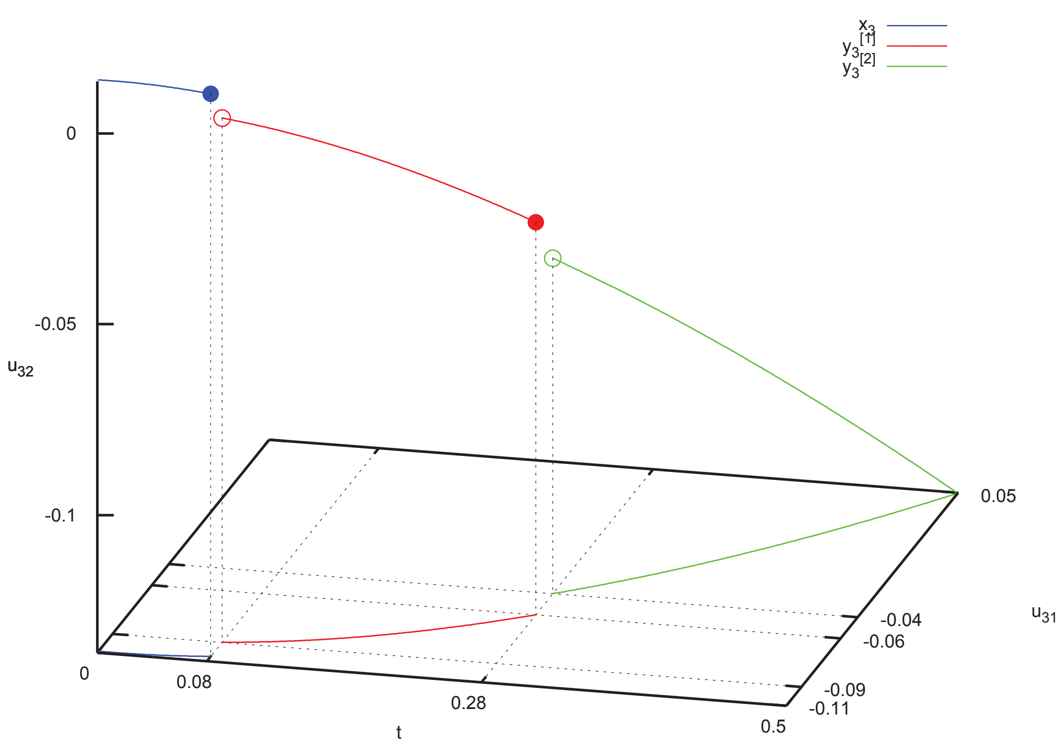

Figure 1. Third approximation (10.7) of a solution of problem (10.1), (10.2), (10.4), (10.5).

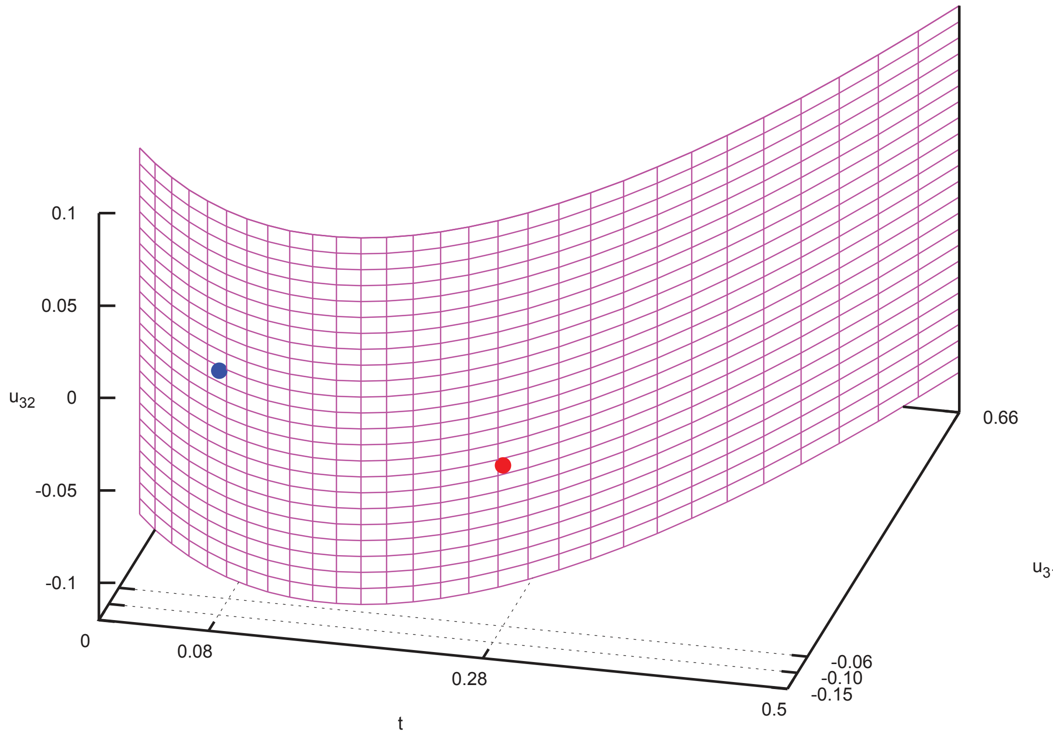

Figure 2. Barrier (10.3) and its points of intersection with function (10.7).

Substituting approximation (10.7) into system (10.1), we obtain residuals estimated as follows:

$$\max_{t\in[0,\hat\tau_1]} \left|X_{31}'(t) - X_{32}^2(t) + X_{31}^2(t) - t\right| \approx 6\cdot 10^{-10},$$
$$\max_{t\in[0,\hat\tau_1]} \left|X_{32}'(t) - X_{31}^2(t) + X_{32}^2(t) + t\right| \approx 3\cdot 10^{-10}$$

for the pre-jump part and, on the time intervals where the jumps occur,

$$\max_{t\in J_k} \left|\frac{\mathrm{d}Y_{31}^{[k]}(t)}{\mathrm{d}t} - (Y_{32}^{[k]}(t))^2 + (Y_{31}^{[k]}(t))^2 - t\right| \approx 10^{-8},$$
$$\max_{t\in J_k} \left|\frac{\mathrm{d}Y_{32}^{[k]}(t)}{\mathrm{d}t} - (Y_{31}^{[k]}(t))^2 + (Y_{32}^{[k]}(t))^2 + t\right| \approx 10^{-8}, \quad k = 1, 2,$$

where $J_1 := [\hat\tau_1, \hat\tau_2]$ and $J_2 := [\hat\tau_1, 0.5]$.

10.2 Barrier (10.3) and one jump

As has already been noted above, a problem of the kind specified may have multiple solutions with different numbers of jumps. We can see this, in particular, on the example of system (10.1), (10.2) with barrier (10.3), for which a two-jump solution has been detected in Section 10.1.

Indeed, consider system (10.1) on $[0, 0.5]$ with the boundary conditions (10.2) and barrier (10.3) and look for a solution with just one jump. In this case, we have $p = 1$ and the state-dependent jump condition at a single unknown point τ_1

$$u_1(\tau_1+) - u_1(\tau_1) = 0.01, \quad u_2(\tau_1+) - u_2(\tau_1) = -0.01, \tag{10.8}$$

where, by (1.6), τ_1 is such that

$$\begin{aligned} u_1(\tau_1) - 7.233\bar{3}\tau_1 + 2.368\bar{3}\tau_1 - 0.04 &= 0,\\ u_1(t) - 7.233\bar{3}t^2 + 2.368\bar{3}t - 0.04 &\neq 0, \quad t \in [0, 0.5] \setminus \{\tau_1\}. \end{aligned} \tag{10.9}$$

Let us find out whether there is such a solution. In contrast to Section 10.1, instead of problem (10.1), (10.2), (10.4), (10.5), we now have problem (10.1), (10.2) (10.8), (10.9).

Calculation of approximate roots of the corresponding determining system (7.1) with $p = 1$ yields for $m = 3$ the third frozen parameters of problem (10.1), (10.2), (10.8), (10.9): $\hat{\tau}_1 = 0.4513818462$ and

$$\begin{aligned} \hat{\xi}_1 &= 0.3957567658, & \hat{\xi}_2 &= -0.2062956437, \\ \hat{\lambda}_1^{[1]} &= 0.4447369584, & \hat{\lambda}_2^{[1]} &= -0.2552758362, \\ \hat{\lambda}_1^{[2]} &= 0.4710845398, & \hat{\lambda}_2^{[2]} &= -0.2816234176. \end{aligned}$$

Using the third frozen parameters, put

$$\begin{aligned} X_3(t) &:= \hat{x}_3(t; \hat{\tau}_1, \hat{\xi}, \hat{\lambda}^{[1]}), & t &\in [a, \hat{\tau}_1], \\ Y_3^{[1]}(t) &:= \hat{y}_3^{[1]}(t; \hat{\tau}_1, 0.5, \hat{\lambda}^{[1]}, \lambda^{[2]}), & t &\in [\hat{\tau}_1, 0.5], \end{aligned}$$

and show that condition (8.6) with $p = 1$ holds for $m = 3$. Consequently, the vector function

$$\hat{u}_3(t) = \begin{cases} X_3(t) & \text{if } t \in [0, \hat{\tau}_1], \\ Y_3^{[1]}(t) & \text{if } t \in (\hat{\tau}_1, 0.5], \end{cases}$$

is the third approximation to a solution of problem (10.1), (10.2), (10.8), (10.9).

10.3 Barrier (10.10) and three jumps

We apply our technique to the same system (10.1) on the interval $[0, 0.5]$ with the same boundary conditions (10.2) but with a different barrier and three jumps allowed. Thus, put $p = 3$, choose the barrier

$$\begin{aligned} G = \{(t, x) \in [0, 0.5] \times \mathbb{R}^2 : x_2 &+ 474.9999931t^4 - 476.6666597t^3 \\ &+ 147.2499979t^2 - 14.43333319t + 0.2 = 0\}, \end{aligned} \tag{10.10}$$

and consider the state-dependent impulse conditions at three unknown points τ_1, τ_2 and τ_3

$$\begin{aligned} u_1(\tau_1+) - u_1(\tau_1) &= 0.01, & u_2(\tau_1+) - u_2(\tau_1) &= -0.01, \\ u_1(\tau_2+) - u_1(\tau_2) &= 0.015, & u_2(\tau_2+) - u_2(\tau_2) &= -0.015, \\ u_1(\tau_3+) - u_1(\tau_3) &= -0.0012, & u_2(\tau_3+) - u_2(\tau_3) &= 0.0012, \end{aligned} \tag{10.11}$$

where, according to (1.6), the time instants τ_1, τ_2 and τ_3 should be such that

$$\begin{aligned} u_2(\tau_k) &+ 474.9999931\tau_k^4 - 476.6666597\tau_k^3 + 147.2499979\tau_k^2 \\ &- 14.43333319\tau_k + 0.2 = 0, \quad k = 1, 2, 3, \\ u_2(t) &+ 474.9999931t^4 - 476.6666597t^3 + 147.2499979t^2 \\ &- 14.43333319t + 0.2 \neq 0, \quad t \in [0, 0.5] \setminus \{\tau_1, \tau_2, \tau_3\}. \end{aligned} \tag{10.12}$$

We are interested in a solution of problem (10.1), (10.2), (10.11), (10.12) as defined in Definition 1.1 with $n = 2$ and $p = 3$. In this case, γ_{τ_1}, γ_{τ_2} and γ_{τ_3} in (1.7) are constant vectors given by the equalities

$$\gamma_{\tau_1} = \operatorname{col}(0.01, -0.01), \quad \gamma_{\tau_2} = \operatorname{col}(0.015, -0.015), \quad \gamma_{\tau_3} = \operatorname{col}(-0.0012, 0.0012) \tag{10.13}$$

and the barrier function g has the form

$$g(t, x) = x_2 + 474.9999931t^4 - 476.6666597t^3 + 147.2499979t^2 - 14.43333319t + 0.2.$$

Introduce the zeroth iterations x_0, $y_0^{[k]}$, $k = 1, 2, 3$, solve system (9.1) of thirteen scalar algebraic equations, and obtain the roots $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3, \hat{\xi}_1, \hat{\xi}_2$ and $\hat{\lambda}_1^{[1]}, \hat{\lambda}_2^{[1]}$, $k = 1, \ldots, 4$, presented in the first column of Table 3. Since, according to Table 3, we have

$$\hat{\xi}_1 = -0.1032363917, \quad \hat{\xi}_2 = 0.01351486331,$$

	$m = 0$	$m = 1$
$\hat{\tau}_1$	0.01786682459	0.01786194368
$\hat{\tau}_2$	0.1570192449	0.1570251944
$\hat{\tau}_3$	0.3110673961	0.3110731609
$\hat{\xi}_1$	−0.1032363917	−0.1025716094
$\hat{\xi}_2$	0.01351486331	0.01347022529
$\hat{\lambda}_1^{[1]}$	−0.103263981	−0.102596893
$\hat{\lambda}_2^{[1]}$	0.01354245262	0.01349550897
$\hat{\lambda}_1^{[2]}$	−0.08216614009	−0.08152274765
$\hat{\lambda}_2^{[2]}$	−0.007555388281	−0.007578636417
$\hat{\lambda}_1^{[3]}$	−0.03123212422	−0.03063507088
$\hat{\lambda}_2^{[3]}$	−0.05848940415	−0.05846631319
$\hat{\lambda}_1^{[4]}$	0.04593622419	0.04641955097
$\hat{\lambda}_2^{[4]}$	−0.1356577526	−0.135520935

Table 3. Frozen parameters for problem (10.1), (10.2), (10.11), (10.12): $m = 0, 1$.

	$m = 2$	$m = 3$
$\hat{\tau}_1$	0.01786194368	0.01786194376
$\hat{\tau}_2$	0.1570251944	0.1570251942
$\hat{\tau}_3$	0.3110731609	0.3110731609
$\hat{\xi}_1$	−0.1025716097	−0.1025716097
$\hat{\xi}_2$	0.01347022531	0.01347022531
$\hat{\lambda}_1^{[1]}$	−0.1025968934	−0.1025968934
$\hat{\lambda}_2^{[1]}$	0.01349550899	0.01349550899
$\hat{\lambda}_1^{[2]}$	−0.08152274798	−0.08152274798
$\hat{\lambda}_2^{[2]}$	−0.007578636385	−0.007578636385
$\hat{\lambda}_1^{[3]}$	−0.03063507121	−0.03063507121
$\hat{\lambda}_2^{[3]}$	−0.05846631315	−0.05846631315
$\hat{\lambda}_1^{[4]}$	0.04641955065	0.04641955065
$\hat{\lambda}_2^{[4]}$	−0.135520935	−0.135520935

Table 4. Frozen parameters for problem (10.1), (10.2), (10.11), (10.12): $m = 2, 3$.

we can take, e.g.,

$$\Omega_0 = [-0.14, 0.04] \times [-0.18, 0.03] = \Omega_1,$$

and then, by (8.1), (8.2), (10.13), we obtain

$$\begin{aligned}\Omega_1^+ &= [-0.15, 0.05] \times [-0.19, 0.04] = \Omega_2,\\ \Omega_2^+ &= [-0.165, 0.065] \times [-0.205, 0.055] = \Omega_3,\\ \Omega_3^+ &= [-0.265, 0.165] \times [-0.305, 0.155] = \Omega_4.\end{aligned}$$

Now choose the vectors

$$\begin{aligned}\varrho^{[0]} &= \mathrm{col}(0.1, 0.1), & \varrho^{[1]} &= \mathrm{col}(0.15, 0.15),\\ \varrho^{[2]} &= \mathrm{col}(0.1, 0.15), & \varrho^{[3]} &= \mathrm{col}(0.1, 0.1)\end{aligned}$$

and, using (8.3), construct the sets

$$\begin{aligned}\mathcal{U}_0 &= [-0.24, 0.14] \times [-0.28, 0.13],\\ \mathcal{U}_1 &= [-0.30, 0.20] \times [-0.34, 0.19],\\ \mathcal{U}_2 &= [-0.265, 0.165] \times [-0.355, 0.205],\\ \mathcal{U}_3 &= [-0.365, 0.265] \times [-0.405, 0.255].\end{aligned}$$

Carrying out computations in Maple, we find that the conditions of Theorems 4.2 and 4.4 are fulfilled and hence the frozen parameter scheme from Section 8 can be applied. The approximate values of roots of (7.1) are presented in Tables 3 and 4.

In addition, computations show that condition (8.6) with $p = 3$ is satisfied for $m = 3$. Consequently, the function

$$\hat{u}_3(t) = \begin{cases} X_3(t) & \text{if } t \in [0, \hat{\tau}_1],\\ Y_3^{[1]}(t) & \text{if } t \in (\hat{\tau}_1, \hat{\tau}_2],\\ Y_3^{[2]}(t) & \text{if } t \in (\hat{\tau}_2, \hat{\tau}_3],\\ Y_3^{[3]}(t) & \text{if } t \in (\hat{\tau}_3, 0.5],\end{cases} \tag{10.14}$$

is the third approximation of a solution of problem (10.1), (10.2), (10.11), (10.12).

The graph of the third approximation $\hat{u}_3$ of a solution of problem (10.1), (10.2), (10.11), (10.12) and its orthogonal projection onto the (t, u_1) plane are given in Figure 3. Figure 4 shows the graph of barrier (10.10) and the three points where it meets the graph of $\hat{u}_3$.

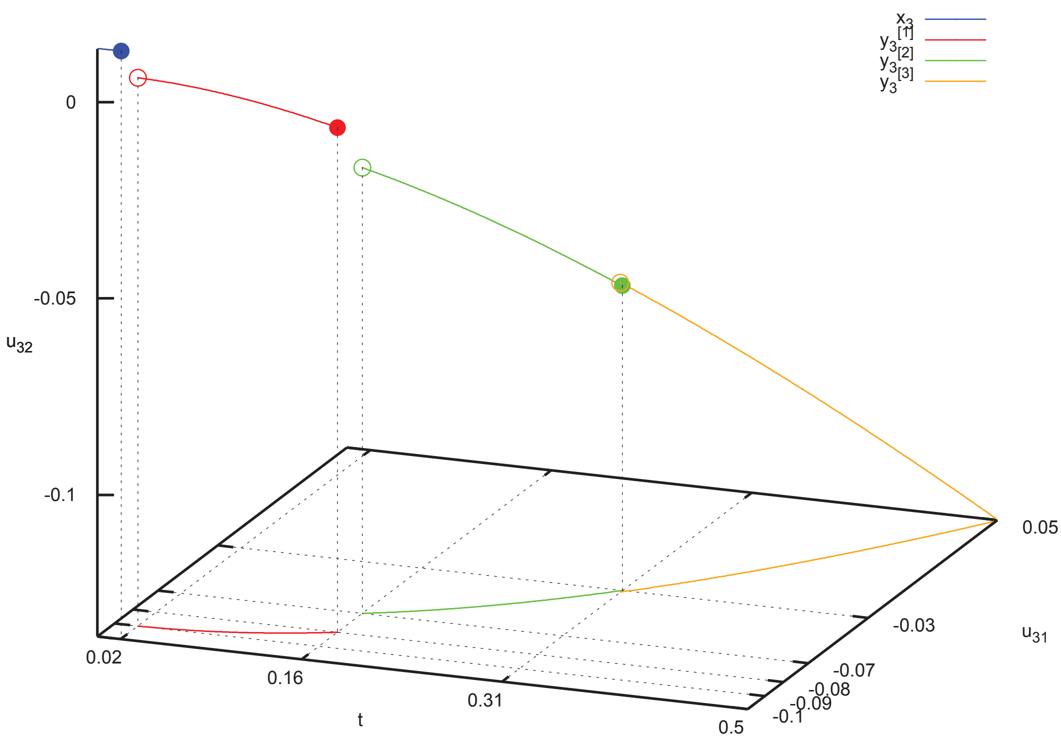

Figure 3. Third approximation (10.14) of a solution of problem (10.1), (10.2), (10.11), (10.12).

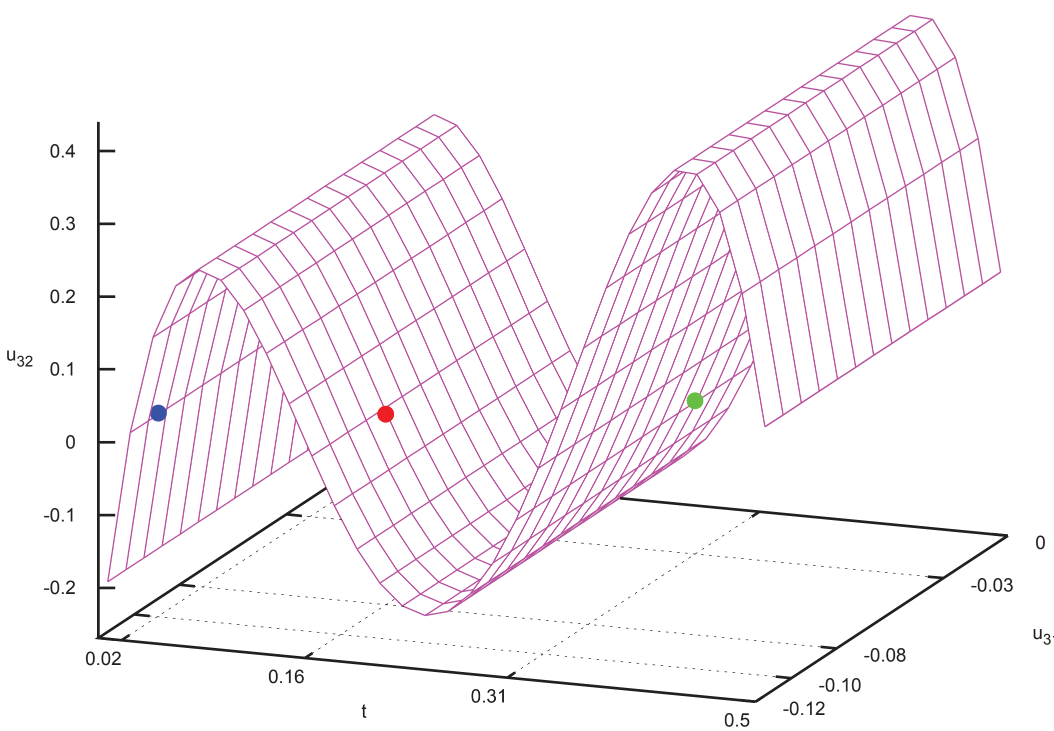

Figure 4. Barrier (10.10) and its points of intersection with function (10.14).

Substituting approximation (10.14) into system (10.1), we obtain a residual estimated as follows:

$$\max_{t\in[a,\hat{\tau}_1]} \left|X'_{31}(t) - (X_{32}(t))^2 + (X_{31}(t))^2 - t\right| \approx 2\cdot 10^{-9},$$
$$\max_{t\in[0,\hat{\tau}_1]} \left|X'_{32}(t) - (X_{31}(t))^2 + (X_{32}(t))^2 + t\right| \approx 2\cdot 10^{-9}$$

and

$$\max_{t\in[\hat{\tau}_1,\hat{\tau}_2]} R_1^{[1]}(t) \approx 8\cdot 10^{-7}, \qquad \max_{t\in[\hat{\tau}_1,\hat{\tau}_2]} R_2^{[1]}(t) \approx 8\cdot 10^{-7},$$
$$\max_{t\in[\hat{\tau}_2,\hat{\tau}_3]} R_1^{[2]}(t) \approx 9\cdot 10^{-7}, \qquad \max_{t\in[\hat{\tau}_2,\hat{\tau}_3]} R_2^{[2]}(t) \approx 9\cdot 10^{-7},$$
$$\max_{t\in[\hat{\tau}_3,b]} R_1^{[3]}(t) \approx 1.5\cdot 10^{-6}, \qquad \max_{t\in[\hat{\tau}_3,b]} R_2^{[3]}(t) \approx 1.5\cdot 10^{-6}.$$

where

$$R_1^{[k]}(t) := \left|\frac{\mathrm{d}Y_{31}^{[k]}(t)}{\mathrm{d}t} - (Y_{32}^{[k]}(t))^2 + (Y_{31}^{[k]}(t))^2 - t\right|,$$
$$R_2^{[k]}(t) := \left|\frac{\mathrm{d}Y_{32}^{[k]}(t)}{\mathrm{d}t} - (Y_{31}^{[k]}(t))^2 + (Y_{32}^{[k]}(t))^2 + t\right|$$

for $k = 1, 2, 3$.

10.4 Barrier (10.15) and two jumps

Let us apply our technique to system (10.1), (10.2) with the barrier

$$G = \{(t, x) \in [0, 0.5] \times \mathbb{R}^2 : x_1^2 + x_2^2 - 0.125t = 0\}, \tag{10.15}$$

and two jumps. More precisely, we put $p = 2$, and pose the state-dependent impulse conditions at two unknown points τ_1 and τ_2

$$\begin{aligned} u_1(\tau_1+) - u_1(\tau_1) &= -0.015625, \quad u_2(\tau_1+) - u_2(\tau_1) = 0.015625,\\ u_1(\tau_2+) - u_1(\tau_2) &= 0.140625, \quad u_2(\tau_2+) - u_2(\tau_2) = -0.140625, \end{aligned} \tag{10.16}$$

where, according to (1.6), the time instants τ_1 and τ_2 should satisfy the conditions

$$\begin{aligned} u_1^2(\tau_k) + u_2^2(\tau_k) - 0.125\,\tau_k &= 0, \quad k = 1, 2,\\ u_1^2(t) + u_2^2(t) - 0.125\,t &\neq 0, \quad t \in [0, 0.5] \setminus \{\tau_1, \tau_2\}. \end{aligned} \tag{10.17}$$

Computation of approximate roots of the corresponding determining system (7.1) for $m = 3$ yields the frozen parameters for problem (10.1), (10.2), (10.16), (10.17):

$$\hat{\tau}_1 = 0.4300565098, \qquad \hat{\tau}_2 = 0.4516205829,$$
$$\hat{\xi}_1 = -0.2868788395, \qquad \hat{\xi}_2 = 0.01090944068,$$
$$\hat{\lambda}_1^{[1]} = -0.2265214409, \qquad \hat{\lambda}_2^{[1]} = -0.0494479579,$$
$$\hat{\lambda}_1^{[2]} = -0.2338309782, \qquad \hat{\lambda}_2^{[2]} = -0.04213842058,$$
$$\hat{\lambda}_1^{[3]} = -0.06866993025, \qquad \hat{\lambda}_2^{[3]} = -0.2072994685.$$

In Figure 5, we present the corresponding third approximation $\hat{u}_3$ of a solution of problem (10.1), (10.2), (10.16), (10.17) and its orthogonal projection onto the (t, u_1) plane. Figure 6 shows the graph of barrier (10.15) and the two points where it is met by the graph of $\hat{u}_3$.

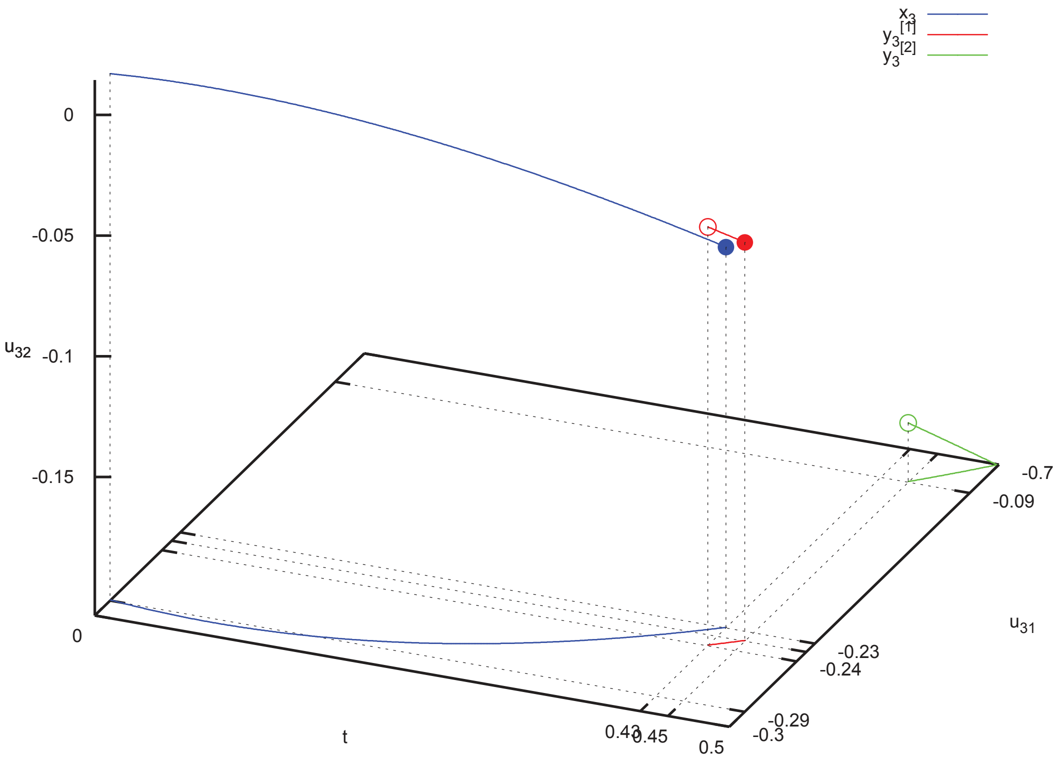

Figure 5. Third approximation $\hat{u}_3$ of a solution of problem (10.1), (10.2), (10.16), (10.17).

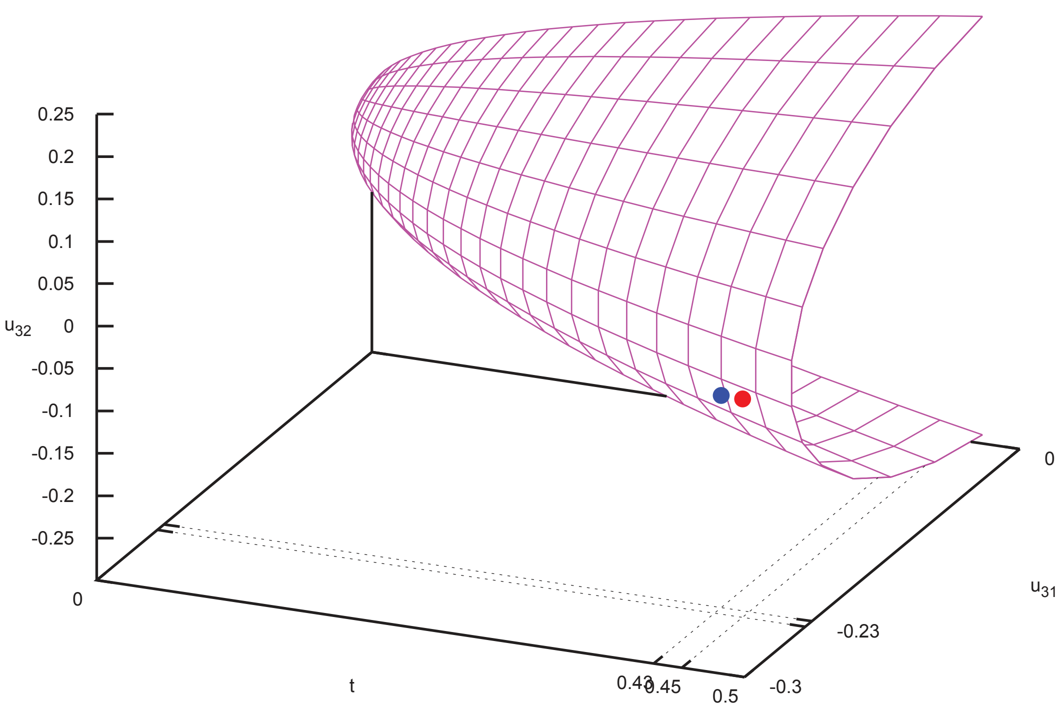

Figure 6. Barrier (10.15) and its points of intersection with $\hat{u}_3$ constructed for problem (10.1), (10.2), (10.16), (10.17).

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ON SOLUTIONS OF NONLINEAR BOUNDARY-VALUE PROBLEMS WHOSE COMPONENTS VANISH AT CERTAIN POINTS

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We show how an appropriate parametrization technique and successive approximations can help to investigate nonlinear boundary-value problems for systems of differential equations under the condition that the components of solutions vanish at certain unknown points. The technique can be applied to nonlinearities involving the signs of the absolute value and positive or negative parts of functions under boundary conditions of various types.

1. Introduction and Problem Setting

The problem of finding solutions of nonlinear differential equations with prescribed numbers of zeros inside a given interval is of interest from numerous points of view (see, e.g., [1, 3–5, 11], and the references therein). This is a rather complicated problem and its investigation is generally based on considerations of pure qualitative nature, which usually do not provide a way to obtain approximations to the analyzed solutions. Further difficulties arise when the equation is studied under nonlinear boundary conditions.

The aim of the present paper is to show that this problem can be efficiently treated by subsequent extensions of the numerical-analytic techniques based on successive approximations suggested for the first time by A. M. Samoilenko [20, 21] for the periodic problem. In the present work, based on the schemes with interval divisions developed in [12–15, 17], we construct a suitable version of this approach for finding solutions with a given number of zeros.

We focus our attention on the system of n nonlinear ordinary differential equations

$$u'(t) = f\big(t, [u(t)]_+, [u(t)]_-\big), \quad t \in [a, b], \tag{1.1}$$

where $[u]_\pm$ for any $u = \operatorname{col}(u_1, \ldots, u_n)$ stands for the vector $\operatorname{col}([u_1]_\pm, \ldots, [u_n]_\pm)$ and $[s]_+ := \max\{s, 0\}$, $[s]_- := \max\{-s, 0\}$ for any real s. System (1.1) is studied under nonlinear two-point boundary conditions of the general form

$$g(u(a), u(b)) = d. \tag{1.2}$$

The functions $f\colon [a, b] \times \Omega \times \Omega \to \mathbb{R}^n$, $g\colon \Omega \times \Omega \to \mathbb{R}^n$ are assumed to be continuous in their domain of definition and the choice of $\Omega \subset \mathbb{R}^n$ is concretized in what follows. Since $[u]_+ - [u]_- = u$ and $[u]_+ + [u]_- = |u|$, system (1.1) includes, e.g., the Fučík-type equations

$$u''(t) = \alpha(t)[u(t)]_+ + \beta(t)[u(t)]_- + q(t, u(t)), \quad t \in [a, b], \tag{1.3}$$

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equations of the Emden–Fowler type

$$u''(t) = p(t)\,|u(t)|^{\lambda}\,u(t) + r(t), \quad t \in [a,b], \tag{1.4}$$

and various other systems of the form

$$u'(t) = h(t, u(t), |u(t)|), \quad t \in [a,b]\,.$$

In what follows, we seek continuously differentiable solutions $u = \operatorname{col}(u_1, u_2, \dots, u_n)$ of Eqs. (1.1), (1.2) each component of which vanishes at some point from (a,b) and has prescribed signs around this point (see Section 2). The numerical-analytic approach [6, 8–11, 18] allows one to approximate these solutions of problem (1.1), (1.2) and, moreover, rigorously prove their existence by using the results of computations [7, 16].

The form of system (1.1) is motivated, in particular, by equations of type (1.3), (1.4), where the terms of form $[u]_{\pm}$, $|u|$ lead to additional difficulties in the practical realization of our scheme due to the necessity of analytic integration of expressions depending on multiple parameters. We show that, in the case of solutions of the indicated kind, the construction of approximations is simplified and any additional approximation of the integrands may be not necessary.

2. Solutions with Fixed Signs on Subintervals

For the convenience of notation, we introduce two definitions (cf. [11]).

Definition 2.1. *Let $\{\sigma_0, \sigma_1\} \subset \{-1, 1\}$ and let t_1 be a point from (a,b). We say that a function $u\colon [a,b] \to \mathbb{R}$ is of type $(\sigma_0, \sigma_1; t_1)$ if $u(t_1) = 0$ and*

$$\sigma_{k-1} u(t) > 0 \qquad \textit{for} \quad t \in (t_{k-1}, t_k), \quad k = 1, 2,$$

where $t_0 := a,\ t_2 := b$.

Suppose that $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \dots, n\} \subset \{-1, 1\}$ and $t_1, t_2, \dots, t_n$ are such that

$$a =: t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} := b. \tag{2.1}$$

Definition 2.2. *We say that a vector function $u = \operatorname{col}(u_1, u_2, \dots, u_n) : [a,b] \to \mathbb{R}^n$ is of type*

$$\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2),\ \dots,\ (\sigma_{n0}, \sigma_{n1}; t_n)\big]$$

if every $u_k,\ k = 1, 2, \dots, n,$ is of type $(\sigma_{k0}, \sigma_{k1}; t_k)$.

In what follows, we seek the solutions of (1.1) possessing the last mentioned property for certain $t_1, t_2, \dots, t_n$. Assumption (2.1) on the order of zeros does not restrict the generality because the equations in (1.1) can always be accordingly renumbered.

Prior to applying the iterative techniques to finding solutions of this kind, it is convenient to simplify the terms involving the positive and negative parts of a function by using the available information on its sign. For this purpose, we set

$$j_\sigma := \frac{1}{2}(\sigma + 1) \tag{2.2}$$

for any $\sigma \in \{-1, 1\}$ and define a function $\tilde{f} : [a, b] \times D \to \mathbb{R}^n$ by setting

$$\tilde{f}(t, u_1, \ldots, u_n) := f\big(t, j_{\sigma_{11}} u_1, \ldots, j_{\sigma_{k-1,1}} u_{k-1}, j_{\sigma_{k0}} u_k, j_{\sigma_{k+1,0}} u_{k+1}, \ldots, j_{\sigma_{n0}} u_n,$$
$$- j_{-\sigma_{11}} u_1, \ldots, -j_{-\sigma_{k-1,1}} u_{k-1}, -j_{-\sigma_{k0}} u_k, -j_{-\sigma_{k+1,0}} u_{k+1}, \ldots, -j_{-\sigma_{n0}} u_n\big) \tag{2.3}$$

for $u = (u_i)_i^n$ from Ω and $t \in [t_{k-1}, t_k],\ k = 1, 2, \ldots, n+1.$

Lemma 2.1. *Let* $\{\sigma_{i0}, \sigma_{i1} :\ i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ *be fixed. Any* $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *solution of Eq. (1.1) is a solution of the system*

$$u'(t) = \tilde{f}(t, u(t)), \quad t \in [a, b], \tag{2.4}$$

where $\tilde{f}$ *is given by (2.3). Conversely, any* $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *solution of Eq. (2.4) satisfies (1.1).*

Proof. Let $u = (u_i)_{i=1}^n$ be a solution of Eq. (1.1) of the type

$$\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots (\sigma_{n0}, \sigma_{n1}; t_n)\big].$$

Since (2.1) is assumed on $t_1, t_2, \ldots, t_n$, it follows from the definition that

$$\operatorname{sign} u_i(t) = s_{ik}, \qquad t \in (t_{k-1}, t_k), \quad k = 1, 2, \ldots, n+1, \tag{2.5}$$

where $S = (s_{ik}),\ i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n+1,$

$$S := \begin{pmatrix} \sigma_{10} & \sigma_{11} & \sigma_{11} & \ldots & \sigma_{11} & \sigma_{11} \\ \sigma_{20} & \sigma_{20} & \sigma_{21} & \ldots & \sigma_{21} & \sigma_{21} \\ \sigma_{10} & \sigma_{30} & \sigma_{30} & \ldots & \sigma_{31} & \sigma_{31} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_{n-1,0} & \sigma_{n-1,0} & \sigma_{n-1,0} & \ldots & \sigma_{n-1,1} & \sigma_{n-1,1} \\ \sigma_{n0} & \sigma_{n0} & \sigma_{n0} & \ldots & \sigma_{n0} & \sigma_{n1} \end{pmatrix}.$$

By (2.2), we get

$$j_{\sigma_{ij}} = (\sigma_{ij} + 1)/2, \quad -j_{-\sigma_{ij}} = (\sigma_{ij} - 1)/2.$$

Together with (2.5), this implies that u satisfies (2.4). The converse implication is obvious.

Lemma 2.1 is proved.

Note that, unlike (1.1), the expression on the right-hand side of the new system (2.4) does not contain positive or negative parts of a function: instead of $[u_i]_+$ and $[u_i]_-$, we find there either u_i, or $-u_i$, or 0, depending on the considered subinterval.

The construction of $\tilde{f}$ is rather easy and proceeds by changing the relevant terms in (1.1) according to their signs. Namely, all occurrences of $[u_i(t)]_+$ in (1.1) are replaced by $u_i(t)$ if $t \in [a, t_i]$, $\sigma_{i0} = 1$ or $t \in (t_i, b]$, $\sigma_{i1} = 1$, and by 0 in all other cases. Similarly, the term $[u_i(t)]_-$ is replaced by $-u_i(t)$ if $t \in [a, t_i]$, $\sigma_{i0} = -1$ or $t \in (t_i, b]$, $\sigma_{i1} = -1$, and by 0, otherwise. Thus, if system (1.1) has the form

$$\begin{aligned} u_1'(t) &= p_{11}(t)[u_1(t)]_+ + p_{12}(t)[u_1(t)]_- + q_1(u_1(t), |u_2(t)|), \\ u_2'(t) &= p_{21}(t)[u_2(t)]_+ + p_{22}(t)[u_2(t)]_- + q_2(u_1(t), u_2(t)), \quad t \in [a, b], \end{aligned} \tag{2.6}$$

and we take, e.g., $\sigma_{10} = 1$, $\sigma_{11} = -1$, $\sigma_{20} = -1$, and $\sigma_{21} = 1$, then the corresponding system (2.4) can be written as follows:

$$\begin{aligned} u_1'(t) &= p_{11}(t)\, u_1(t) + q_1(u_1(t), -u_2(t)), \\ u_2'(t) &= -p_{22}(t)\, u_2(t) + q_2(u_1(t), u_2(t)) \end{aligned} \tag{2.7}$$

for $t \in [a, t_1]$,

$$\begin{aligned} u_1'(t) &= -p_{12}(t)\, u_1(t) + q_1(u_1(t), -u_2(t)), \\ u_2'(t) &= -p_{22}(t)\, u_2(t) + q_2(u_1(t), u_2(t)) \end{aligned} \tag{2.8}$$

for $t \in [t_1, t_2]$, and

$$\begin{aligned} u_1'(t) &= -p_{12}(t)\, u_1(t) + q_1(u_1(t), u_2(t)), \\ u_2'(t) &= p_{21}(t)\, u_2(t)_- + q_2(u_1(t), u_2(t)) \end{aligned} \tag{2.9}$$

for $t \in [t_2, b]$ (recall that $a < t_1 < t_2 < b$). In this case, the assertion of Lemma 2.1 means that if we restrict our consideration to solutions $u = \operatorname{col}(u_1, u_2)$ of the type $[(1, -1; t_1), (-1, 1; t_2)]$ in a sense of Definition 2.2, then the original system (2.6) can be equivalently rewritten on the relevant subintervals as (2.7)–(2.9).

Remark 2.1. It is not difficult to verify that formula (2.3) for $\tilde{f}$ can be alternatively represented as

$$\tilde{f}(t, u) = f\left(t, \frac{1}{2}(M_k + I)u(t), \frac{1}{2}(M_k - I)u(t)\right), \tag{2.10}$$

where $u = (u_i)_i^n$ is from Ω, $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n+1$, I is the unit matrix, and

$$M_k := \operatorname{diag}\left(\sigma_{11}, \sigma_{21}, \ldots, \sigma_{k-1,1}, \sigma_{k0}, \sigma_{k+1,0}, \ldots, \sigma_{n0}\right). \tag{2.11}$$

Equality (2.10) implies, in particular, that possible occurrences of $|u_i|$ in the original system are replaced by the ith component of $M_k u$ in $\tilde{f}$ on $[t_{k-1}, t_k]$.

By using Remark 2.1 in the example presented above, we can easily get system (2.7)–(2.9) on the three intervals directly because, in this case, in view of (2.11), the matrices $M_1 = \operatorname{diag}(\sigma_{10}, \sigma_{20})$, $M_2 = \operatorname{diag}(\sigma_{11}, \sigma_{20})$, and $M_3 = \operatorname{diag}(\sigma_{11}, \sigma_{21})$ take the form

$$M_1 = \operatorname{diag}(1, -1), \qquad M_2 = \operatorname{diag}(-1, -1), \qquad \text{and} \qquad M_3 = (-1, 1).$$

Therefore, on $[t_{k-1}, t_k], 1 \le k \le 3$, the occurrences of $[u_i]_+$ (resp., $[u_i]_-$) in (2.6) are replaced by $(1/2)[(M_k u)_i + u_i]$ (resp., $(1/2)[(M_k u)_i - u_i]$), and $|u_2|$ is replaced by $(M_k u)_2$.

3. Parametrization and Auxiliary Problems

We fix some $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ and focus on finding the solutions of (1.1) of the type $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ for *some* $t_1, t_2, \ldots, t_n$ from (a, b). The values of $t_1, t_2, \ldots, t_n$ are *a priori* unknown and should be determined parallel with u. Without loss of generality we assume that these points are ordered as indicated in (2.1).

The idea that is used in what follows suggests to replace the boundary-value problem (2.4), (1.2) by a suitable family of "model-type" problems with separated boundary conditions. The construction of these problems is very simple. We "freeze" the values of $u = \mathrm{col}(u_1, u_2, \ldots, u_n)$ at points (2.1) by (formally) setting

$$u(t_k) = z^{(k)}, \quad k = 0, 1, \ldots, n+1, \tag{3.1}$$

where

$$z^{(k)} = \mathrm{col}\big(z_1^{(k)}, z_1^{(k)}, \ldots, z_n^{(k)}\big),$$

and consider the restrictions of system (2.4) to each interval $[t_0, t_1], [t_1, t_2], \ldots, [t_n, t_{n+1}]$. This gives $n+1$ two-point boundary-value problems on the respective subintervals

$$u'(t) = \tilde{f}(t, u(t)), \quad t \in [t_{k-1}, t_k], \tag{3.2}$$

$$u(t_{k-1}) = z^{(k-1)}, \quad u(t_k) = z^{(k)}, \tag{3.3}$$

where $k = 1, 2, \ldots, n+1$, and $\tilde{f}$ is given by (2.3). We fix certain nonempty bounded sets

$$\Omega_k \subset \mathbb{R}^n, \quad k = 0, 1, \ldots, n+1, \tag{3.4}$$

and treat the vectors $z^{(j)}$ appearing in (3.1), (3.3), and (3.4) as parameters with values in Ω_j, $j = 0, 1, \ldots, n+1$. We use the family of problems (3.2), (3.3) to study the $[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)]$ solutions $u = \mathrm{col}(u_1, u_2, \ldots, u_n)$ of problem (1.1), (1.2) whose values at the unknown points (2.1) lie in the corresponding sets (3.4), i.e., such that

$$u(t_k) \in \Omega_k, \quad k = 0, 1, 2, \ldots, n+1. \tag{3.5}$$

After simplification of the original system (1.1) by using the sign properties of solutions, we impose restrictions required to apply our method directly to the transformed system (2.4). The conditions on $\tilde{f}$ are imposed on certain sets somewhat wider than the already fixed sets (3.4).

Given sets (3.4), for any $k = 1, 2, \ldots, n+1$, we introduce the sets

$$\Omega_{k-1,k} := \big\{(1-\theta)\xi + \theta\eta : \xi \in \Omega_{k-1},\ \eta \in \Omega_k,\ \theta \in [0, 1]\big\}. \tag{3.6}$$

It is clear that $\Omega_{k-1,k}$ is formed by all possible straight segments joining the points of Ω_{k-1} with the points of Ω_k. Further, we also need the componentwise $\varrho^{(k)}$-neighborhoods of $\Omega_{k-1,k}$, $k = 1, \ldots, n+1$:

$$\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}), \quad k = 1, 2, \ldots, n+1, \tag{3.7}$$

where

$$\mathcal{O}_\varrho(\Omega) := \bigcup_{\xi\in\Omega} \mathcal{O}_\varrho(\xi) \tag{3.8}$$

and

$$\mathcal{O}_\varrho(\xi) := \{\nu \in \mathbb{R}^n : |\nu - \xi| \le \varrho\}$$

for any $\Omega \subset \mathbb{R}^n$, $\varrho \in \mathbb{R}^n_+$, $\xi \in \Omega$. The values of $\varrho^{(k)}$, $k = 1, 2, \ldots, n+1$, used in (3.7) are chosen in what follows. The conditions formulated in Section 4 are imposed on sets (3.7) with respect to the space variables.

4. Assumptions

To study the solutions of the auxiliary problems (3.2), (3.3) with $z^{(j)} \in \Omega_j$, $j = 0, 1, \ldots, n+1$, we use suitable parametrized successive approximations analytically constructed on the subintervals $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n+1$. Since we seek solutions of type $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ with given $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ and some unknown $t_1, \ldots, t_n$, we assume that the set Ω_0 is chosen so that

$$\Pi_i \Omega_0 \subset \sigma_{i0}\mathbb{R}_+, \quad i = 1, 2, \ldots, n, \tag{4.1}$$

where

$$\Pi_i \Omega := \{s_i : (s_1, \ldots, s_i, \ldots, s_n) \in \Omega \quad \text{for some} \quad s_1, \ldots, s_n\}.$$

Remark 4.1. Due to the nature of the problem under consideration, in addition to (4.1), it is natural to suppose that the sets Ω_0, $\Omega_1, \ldots, \Omega_{n+1}$ have the properties

$$\Pi_i \Omega_j \subset v_{ji}\mathbb{R}_+, \qquad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, n+1, \tag{4.2}$$

where $v_k = (v_{ki})_{i=1}^n$ are defined as follows:

$$v_0 := (\sigma_{10}, \sigma_{20}, \ldots, \sigma_{n0}),$$

$$v_k = (\sigma_{11}, \sigma_{21}, \ldots, \sigma_{k-1,1}, 0, \sigma_{k+1,0}, \sigma_{n0}), \quad k = 1, 2, \ldots, n,$$

$$v_{n+1} := (\sigma_{11}, \sigma_{21}, \ldots, \sigma_{n1}).$$

Although relations (4.2) are useful because they exclude from consideration the sets that cannot contain the values of solutions of the type $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$, in what follows, it is sufficient to impose condition (4.1) fixing the signs of the solution in the initial subinterval.

We now need two assumptions concerning the function $\tilde{f}$ appearing in (2.4) and (3.2).
Assume that there exist nonnegative vectors $\varrho^{(1)}$, $\varrho^{(2)}, \ldots,$ $\varrho^{(n+1)}$ such that

$$\varrho^{(k)} \ge \frac{t_k - t_{k-1}}{4} \delta_{[t_{k-1}, t_k], \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k})}(\tilde{f}) \tag{4.3}$$

for all $k = 1, 2, \ldots, n+1$, where

$$\delta_{[\alpha,\beta],\Omega}(\tilde{f}) := \max_{(t,u)\in[\alpha,\beta]\times\Omega} \tilde{f}(t,u) - \min_{(t,u)\in[\alpha,\beta]\times\Omega} \tilde{f}(t,u) \tag{4.4}$$

for $a < \alpha < \beta < b$ and a closed $\Omega \subset \mathbb{R}^n$.

We fix certain $\varrho^{(1)},\ \varrho^{(2)}, \ldots,\ \varrho^{(n+1)}$ for which (4.3) holds, consider the sets

$$\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}), \quad k = 1, 2, \ldots, n+1,$$

and suppose that, for some nonnegative matrices $K_k,\ k = 1, 2, \ldots, n+1$, the function $\tilde{f}$ satisfies the Lipschitz condition

$$\left|\tilde{f}(t, y_1) - \tilde{f}(t, y_2)\right| \le K_k \left|y_1 - y_2\right| \tag{4.5}$$

for $t \in [t_{k-1}, t_k],\ \{y_1, y_2\} \subset \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}),\ k = 1, 2, \ldots, n+1$. Finally, we assume that

$$r(K_k) < \frac{10}{3(t_k - t_{k-1})} \tag{4.6}$$

for all $k = 1, 2, \ldots, n+1$.

Remark 4.2. In finding the solutions vanishing at certain points (this is the case, in particular, for the class of solutions defined in Section 2), the direct verification of condition (4.6) is impossible because the values of $t_1, t_2, \ldots, t_n$ are unknown. Obviously, the validity of (4.6) is guaranteed if

$$\max_{1\le k\le n+1} r(K_k) < \frac{10}{3(b-a)}. \tag{4.7}$$

It does make sense, however, to keep inequalities (4.6) because they may lead us to conditions much weaker than (4.7) if some estimates for $t_1, t_2, \ldots, t_n$ are available (see Section 6).

Remark 4.3. In order to verify condition (4.3) on $\varrho^{(0)}, \ldots, \varrho^{(n+1)}$, it is necessary to compute the maximal and minimal values of the function $\tilde{f}$ over $\varrho^{(k)}$-neighborhoods of the sets $\Omega_{k-1,k},\ k = 1, 2, \ldots, n+1$, constructed according to (3.6). One may use computer software for this purpose. It is convenient to specify suitable sets

$$\Omega^{(k)} \supset \Omega_{k-1,k}, \quad k = 1, 2, \ldots, n+1, \tag{4.8}$$

with simpler structures (e.g., parallelepipeds: If $\Omega^{(k)}$ is a parallelepiped, then, by (3.8), the set $\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})$ is also a parallelepiped) and use the inequality

$$\delta_{[\alpha,\beta],\tilde{\Omega}}(\tilde{f}) \ge \delta_{[\alpha,\beta],\Omega}(\tilde{f})$$

for any $\tilde{\Omega} \supset \Omega$, which is an immediate consequence of (4.4). Then the validity of (4.3) is guaranteed if

$$\varrho^{(k)} \ge \frac{t_k - t_{k-1}}{4} \delta_{[t_{k-1},t_k],\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})}(\tilde{f}) \tag{4.9}$$

for $k = 1, 2, \ldots, n+1$. The same observation is true for the Lipschitz condition (4.5). It can be easier checked on the set $\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})$ instead of the set $\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}),\ k = 1, 2, \ldots, n+1$.

5. Successive Approximations and Determining Equations

As shown above, the problem of $\big[(\sigma_{10},\sigma_{11};t_1),(\sigma_{20},\sigma_{21};t_2),\dots,(\sigma_{n0},\sigma_{n1};t_n)\big]$ solutions of the boundary-value problem (1.1), (1.2) reduces to the same problem (1.2) for equation (2.4), where the function $\tilde f$ is constructed according to (2.3). To treat problem (2.4), (1.2), we can use the approach proposed in [15, 17] with the use of the properties of the auxiliary problems (3.2) and (3.3). In what follows, we always assume that conditions (4.3), (4.5), and (4.6) are satisfied.

We now define parametrized recurrence sequences of functions $u_m^{(k)}\big(\cdot,z^{(k-1)},z^{(k)},t_{k-1},t_k\big),\ m=0,1,\dots,$ by setting

$$u_0^{(k)}(t,z^{(k-1)},z^{(k)},t_{k-1},t_k):=\left(1-\frac{t-t_{k-1}}{t_k-t_{k-1}}\right)z^{(k-1)}+\frac{t-t_{k-1}}{t_k-t_{k-1}}z^{(k)}, \tag{5.1}$$

$$\begin{aligned} u_m^{(k)}(t,z^{(k-1)},z^{(k)},t_{k-1},t_k) &:= u_0^{(k)}(t,z^{(k-1)},z^{(k)}) \\ &\quad+\int_{t_{k-1}}^{t}\tilde f\big(s,u_{m-1}^{(k)}(s,z^{(k-1)},z^{(k)},t_{k-1},t_k)\big)ds \\ &\quad-\frac{t-t_{k-1}}{t_k-t_{k-1}}\int_{t_{k-1}}^{t_k}\tilde f\big(s,u_{m-1}^{(k)}(s,z^{(k-1)},z^{(k)},t_{k-1},t_k)\big)ds \end{aligned} \tag{5.2}$$

for all $m=1,2,\dots,\ z^{(0)}\in\Omega_0,\ z^{(k)}\in\Omega_k,\ t\in[t_{k-1},t_k],\ k=1,2,\dots,n+1$. Recall that $t_0=a,\ t_{n+1}=b$, while the intermediate times $t_1,\dots,t_n$ are regarded as unknown parameters.

It is clear that every function $u_m^{(k)}(\cdot,z^{(k-1)},z^{(k)},t_{k-1},t_k),\ m=0,1,\dots,$ satisfies conditions (3.3) independently of the choice of $z^{(k-1)}$ and $z^{(k)}$:

$$u_m^{(k)}\big(t_{k-1},z^{(k-1)},z^{(k)},t_{k-1},t_k\big)=z^{(k-1)},\qquad u_m^{(k)}\big(t_k,z^{(k-1)},z^{(k)},t_{k-1},t_k\big)=z^{(k)}. \tag{5.3}$$

The sequences given by (5.1), (5.2) are helpful for the investigation of the auxiliary problems (3.2) and (3.3) and, ultimately, of the given problem (1.1), (1.2).

Theorem 5.1. *Assume that (4.3), (4.5), and (4.6) are satisfied. Then, for any fixed* $z^{(k)}\in\Omega_k,\ k=0,1,\dots,n+1:$

1. *Functions (5.2) are continuously differentiable on* $t\in[t_{k-1},t_k],\ k=1,\dots,n+1,$ *and the inclusion*

$$\Big\{u_m^{(k)}(t,z^{(k-1)},z^{(k)},t_{k-1},t_k):t\in[t_{k-1},t_k]\Big\}\subset\mathcal O_{\varrho^{(k)}}(\Omega_{k-1,k}) \tag{5.4}$$

holds.

2. *The limit*

$$\lim_{m\to\infty}u_m^{(k)}\big(t,z^{(k-1)},z^{(k)},t_{k-1},t_k\big)=:u_\infty^{(k)}\big(t,z^{(k-1)},z^{(k)},t_{k-1},t_k\big)$$

exists uniformly in $t\in[t_{k-1},t_k],\ k=1,2,\dots,n+1.$

3. *The limit functions satisfy the separated two-point boundary conditions*

$$\begin{aligned} u_\infty^{(k)}\big(t_{k-1}, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) &= z^{(k-1)}, \\ u_\infty^{(k)}\big(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) &= z^{(k)}. \end{aligned} \tag{5.5}$$

4. *The function* $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ *is the unique continuously differentiable solution of the integral equation*

$$u(t) = u_0^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) + \int\limits_{t_{k-1}}^{t} \tilde f(s, u(s))ds - \frac{t - t_{k-1}}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} \tilde f(s, u(s))\, ds, \quad t \in [t_{k-1}, t_k], \tag{5.6}$$

with values in $\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k})$.

5. *For any* $m \geq 0,$ *the following estimate is true:*

$$\big|u_\infty^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) - u_m^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)\big| \leq \frac{5}{9}\alpha_1(t, t_{k-1}, t_k)\, Q_k^m\, (I - Q_k)^{-1}\, \delta_{[t_{k-1}, t_k], \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k})}(f),$$

where $Q_k := (3/10)(t_k - t_{k-1})K_k$ *and*

$$\alpha_1(t, t_{k-1}, t_k) := 2\,(t - t_{k-1})\left(1 - \frac{t - t_{k-1}}{t_k - t_{k-1}}\right)$$

for $t \in [t_{k-1}, t_k]$.

It follows from (5.6) that the function $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big),\ k = 1, 2, \ldots, n+1,$ is the unique solution of the Cauchy problem for the system

$$u'(t) = \tilde f(t, u(t)) + \frac{1}{t_k - t_{k-1}} \Delta^{(k)}(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k), \tag{5.7}$$

$$u(t_{k-1}) = z^{(k-1)}, \tag{5.8}$$

where $\Delta^{(k)} \colon \Omega_{k-1} \times \Omega_k \times (a, b)^2 \to \mathbb{R}^n,\ k = 1, \ldots, n+1,$ is given by the formula

$$\Delta^{(k)}(\xi, \eta, s_0, s_1) := \eta - \xi - \int\limits_{t_{k-1}}^{t_k} \tilde f\big(s, u_\infty^{(k)}(s, \xi, \eta, s_0, s_1)\big)ds \tag{5.9}$$

for all $\xi \in \Omega_{k-1},\ \eta \in \Omega_k,$ and $\{s_0, s_1\} \subset (a, b)$.

The proof is realized by analogy with [17] (Theorem 1) and [15] (Theorem 5.1). The starting point is to establish inclusion (5.4).

It is natural to expect that the limit functions

$$u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big), \quad k = 1, 2, \ldots, n+1,$$

of iterations (5.2) on the subintervals $t \in [t_{k-1}, t_k]$ would enable us to formulate the criteria of solvability of the original problem (1.1), (1.2). It turns out that the functions

$$\Delta^{(k)} : \Omega_{k-1} \times \Omega_k \times (a, b)^2 \to \mathbb{R}^n, \quad k = 1, 2, \ldots, n+1, \tag{5.10}$$

defined by equalities (5.9) guarantee the validity of these conclusions. Indeed, Theorem 5.1 guarantees that under the imposed conditions, the functions

$$u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) : [t_{k-1}, t_k] \to \mathbb{R}^n, \quad k = 1, 2, \ldots, n+1,$$

are well defined for all $(z^{(k-1)}, z^{(k)}) \in \Omega_{k-1} \times \Omega_k$ and $(t_{k-1}, t_k) \in (a, b)^2$. Therefore, by setting

$$u_\infty\big(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, t_2, \ldots, t_n\big) := u_\infty^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) \tag{5.11}$$

for $t \in [t_{k-1}, t_k],\ k = 1, 2, \ldots, n+1,$ we obtain a function

$$u_\infty(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n) : [a, b] \to \mathbb{R}^n.$$

This function is obviously continuous at the points $t_k,\ k = 1, 2, \ldots, n,$ because, by virtue of (5.5),

$$u_\infty^{(k)}\big(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = u_\infty^{(k+1)}\big(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\big).$$

Along with (3.2), we consider the equations with constant forcing terms

$$u'(t) = \tilde{f}(t, u(t)) + \frac{1}{t_k - t_{k-1}}\mu^{(k)}, \quad t \in [t_{k-1}, t_k], \tag{5.12}$$

under the initial conditions

$$u(t_{k-1}) = z^{(k-1)}, \tag{5.13}$$

where

$$\mu^{(k)} = \operatorname{col}(\mu_1^{(k)}, \mu_2^{(k)}, \ldots, \mu_n^{(k)}), \quad k = 1, 2, \ldots, n+1,$$

are control parameters. Then, by analogy with [19] (Theorem 2), we obtain the following theorem:

Theorem 5.2. *Assume that (4.3), (4.5), and (4.6) are true. Let $z^{(j)} \in \Omega_j,\ j = 0, 1, \ldots, n+1,$ be fixed. Then, for the solutions of the Cauchy problems (5.12), (5.13) to have the properties*

$$u(t_k) = z^{(k)}, \quad k = 1, 2, \ldots, n+1, \tag{5.14}$$

it is necessary and sufficient that $\mu^{(k)}$ *have the form*

$$\mu^{(k)} = \Delta^{(k)}\big(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big), \quad k = 1, 2, \ldots, n+1. \tag{5.15}$$

In this case, the solution of (5.12), (5.13) coincides with $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ *for any* $k = 1, 2, \ldots, n+1.$

The next statement establishes the relationship between function (5.11) and the solutions of the original problem (1.1), (1.2) in the terms of the zeros of functions (5.10). Recall that Ω_0 is chosen so that (4.1) holds.

Theorem 5.3. *Let (4.3), (4.5), and (4.6) hold. Then the function*

$$u_\infty(\cdot, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n) \colon [a, b] \to \mathbb{R}^n$$

is a continuously differentiable solution of the boundary-value problem (1.1), (1.2) if and only if the vectors $z^{(k)}$, $k = 0, 1, 2, \ldots, n+1,$ *and the points* $t_1, \ldots, t_n$ *satisfy the system of* $n(n+2)$ *numerical determining equations*

$$\Delta^{(k)}\big(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = 0, \quad k = 1, 2, \ldots, n+1, \tag{5.16}$$

$$g\big(u_\infty^{(1)}(a, z^{(0)}, z^{(1)}, a, t_1), u_\infty^{(n+1)}(b, z^{(n)}, z^{(n+1)}, t_n, b)\big) = d. \tag{5.17}$$

Furthermore, this is a solution of the $\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *type.*

Proof. The required statement is proved similarly to [19] (Theorem 3). We use Lemma 2.1 and take into account the choice of the domain Ω_0 according to (4.1), which, by virtue of the unique solvability of the Cauchy problems (5.7), (5.8), excludes the existence of solutions not possessing the prescribed property $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$.

Theorem 5.3 is proved.

Finally, the following analog of [19] (Theorem 4) shows that the determining equations (5.16), (5.17) detect all possible solutions of problem (1.1), (1.2) with graphs lying in the specified domains.

Theorem 5.4. *Let (4.3), (4.5), and (4.6) hold. If there exist some* $t_1, \ldots, t_n$ *from* (a, b) *and* $z^{(j)} \in \Omega_j$, $j = 0, 1, \ldots, n+1,$ *satisfying the determining equations (5.16), (5.17), then the function*

$$u^*(t) = u_\infty\big(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n\big), \quad t \in [a, b], \tag{5.18}$$

is a $\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *solution of the boundary-value problem (1.1), (1.2). Conversely, if problem (1.1), (1.2) possesses a solution* $u^*(\cdot)$ *of the* $\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *type, which, in addition, satisfies the conditions*

$$u^*(t_j) \in \Omega_j, \quad j = 0, 1, \ldots, n+1,$$

$$\big\{u^*(t) \colon t \in [t_{k-1}, t_k]\big\} \subset \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}), \quad k = 1, 2, \ldots, n+1,$$

then the system of determining equations (5.16), (5.17) is satisfied with the same $t_1, \ldots, t_n$ *and*

$$z^{(j)} := u^*(t_j), \quad j = 0, 1, \ldots, n+1.$$

Moreover, the solution $u^*(\cdot)$ *necessarily has form (5.18) with the indicated values of the parameters.*

Remark 5.1. In the case of $\big[(\sigma_{10},\sigma_{11};t_1),\ (\sigma_{20},\sigma_{21};t_2),\dots,(\sigma_{n0},\sigma_{n1};t_n)\big]$ solutions, the parameters $z^{(1)}$, $z^{(2)},\dots,z^{(n)}$ in the auxiliary two-point problems (3.2), (3.3) have the form

$$z^{(k)} = \operatorname{col}\big(z_1^{(k)},\dots,z_{k-1}^{(k)},0,z_{k+1}^{(k)},\dots,z_n^{(k)}\big), \quad k=1,2,\dots,n, \tag{5.19}$$

and, therefore, system (5.16), (5.17) involves $n(n+1)$ variables.

6. Determination of Approximate Solutions

Although Theorem 5.4 gives a theoretical description of all solutions of problem (2.4), (1.2) with graphs contained in a given region, its direct application is difficult because the form of the limit functions of sequences and (5.1), (5.2) is usually unknown and, hence, the determining equations (5.16), (5.17) rarely admit explicit representations. The difficulty can be overcome in a customary way (see, e.g., [6, 12] and the references therein) if we replace the unknown limit $u_\infty^{(k)}\big(\cdot,z^{(k-1)},z^{(k)},t_{k-1},t_k\big)$ in (5.11) by an iteration $u_m^{(k)}\big(\cdot,z^{(k-1)},z^{(k)},t_{k-1},t_k\big)$, $k=1,2,\dots,n+1$, of the form (5.2) for fixed m. In this way, we obtain the function

$$u_m\big(t,z^{(0)},z^{(1)},z^{(2)},\dots,z^{(n+1)},t_1,\dots,t_n\big) := u_m^{(k)}\big(t,z^{(k-1)},z^{(k)},t_{k-1},t_k\big) \tag{6.1}$$

for $t\in[t_{k-1},t_k],\ k=1,2,\dots,n+1$. We see that (6.1) is an approximate version of the unknown function (5.11). Its values can be found explicitly for all values of the parameters. By using function (6.1), we arrive, in a natural way, at the mth approximate system of determining equations

$$\Delta_m^{(k)}\big(z^{(k-1)},z^{(k)},t_{k-1},t_k\big)=0, \quad k=1,2,\dots,n+1, \tag{6.2}$$

$$g\big(u_m^{(1)}\big(a,z^{(0)},z^{(1)},a,t_1\big),u_m^{(n+1)}(b,z^{(n)},z^{(n+1)},t_n,b)\big)=d, \tag{6.3}$$

where, by the direct analogy with (5.9), the functions $\Delta_m^{(k)}:\Omega_{k-1}\times\Omega_k\times(a,b)^2\to\mathbb{R}^n,\ k=1,\dots,n+1$, are defined as follows:

$$\Delta_m^{(k)}(\xi,\eta,s_0,s_1) := \eta-\xi-\int_{t_{k-1}}^{t_k}\tilde f\big(s,u_m^{(k)}(s,\xi,\eta,s_0,s_1)\big)ds \tag{6.4}$$

for $\xi\in\Omega_{k-1},\ \eta\in\Omega_k$, and $\{s_0,s_1\}\subset(a,b)$. Note that, unlike system (5.16), (5.17), the mth approximate system (6.2), (6.3) contains only terms involving the functions $u_m^{(k)}\big(\cdot,z^{(k-1)},z^{(k)},t_{k-1},t_k\big),\ k=1,2,\dots,n+1$, which can be found explicitly.

The approximate solutions of the original problem are, as usual, obtained (see, e.g., [6, 12]) by substituting the roots of the corresponding approximate determining system (6.2), (6.3) in (6.1). According to the approach described in the present work, the approximations are constructed by "gluing" together the curves obtained on each single interval $[t_{k-1},t_k],\ k=1,2,\dots,n+1$. This gluing is smooth.

Lemma 6.1. *If $z^{(k)}\in\Omega_k,\ k=0,1,2,\dots,n+1$, satisfy equations (6.2) for some m, then the corresponding function (6.1) is continuously differentiable on $[a,b]$.*

Proof. We fix $z^{(j)},\ j=0,1,\dots,n+1$, set

$$v := u_m\big(\cdot,z^{(0)},z^{(1)},z^{(2)},\dots,z^{(n+1)},t_1,\dots,t_n\big),$$

and consider the values of v around t_k for fixed $k = 1, 2, \ldots, n.$ By (6.1), it suffices to check only

$$u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) \qquad \text{and} \qquad u_m^{(k+1)}\big(\cdot, z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\big).$$

Indeed, it immediately follows from (5.2) that

$$v'(t_k-) = \tilde{f}\big(t_k, u_{m-1}^{(k)}(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)\big) + \frac{1}{t_k - t_{k-1}}\, \Delta_m^{(k)}\big(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) \tag{6.5}$$

and

$$v'(t_k+) = \tilde{f}\big(t_k, u_{m-1}^{(k+1)}(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1})\big) + \frac{1}{t_{k+1} - t_k}\, \Delta_m^{(k+1)}\big(z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\big). \tag{6.6}$$

Since $z^{(j)},\ j = 0, 1, \ldots, n+1,$ are supposed to satisfy (6.2), equalities (6.5), (6.6) imply that

$$\begin{aligned} v'(t_k-) &= \tilde{f}\big(t_k, x_{m-1}^{(k)}(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)\big), \\ v'(t_k+) &= \tilde{f}\big(t_k, x_{m-1}^{(k+1)}(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1})\big). \end{aligned} \tag{6.7}$$

However, in view of (5.3), we find

$$u_{m-1}^{(k)}\big(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = u_{m-1}^{(k+1)}\big(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\big) = z^{(k)},$$

which, together with (6.7), yields $v'(t_k-) = v'(t_k+)$.

Lemma 6.1 is proved.

The solvability of the determining system (5.16), (5.17) can be studied by analogy with [7, 16] by using the topological degree methods [2] and analyzing some of its approximate versions (6.2), (6.3) (this problem is not considered in the present work).

A special note should be made concerning the verification of the assumptions made in Section 4. Namely, both relations (4.3) that should be satisfied by $\varrho^{(1)},\ \varrho^{(2)}, \ldots, \varrho^{(n+1)}$ and inequalities (4.6) for $r(K_j),\ j = 1, \ldots, n+1,$ depend on the unknown $t_1,\ t_2, \ldots, t_n$. Although one can replace the subintervals by the entire $[a, b]$ [cf. (4.7)], this would lead to more restrictive conditions. Another (better) opportunity is to use the preliminary results of computations performed according to the scheme described above.

Indeed, it is always expedient to start computations directly *prior to* checking conditions (4.3), (4.6) because, in this way, we can obtain preliminary information on the space localization of solutions and, as a consequence, a useful hint how to choose the regions where the conditions should be verified. This concerns both the choice of the sets $\Omega_k,\ k = 0, 1, \ldots, n+1,$ with respect to the space variables and the intervals containing zeros of the solutions.

Suppose that we start direct computations and try to solve approximate determining equations. If the computations show reasonable, in a certain sense, results and we get certain approximate values

$$\hat{t}_1,\ \hat{t}_2,\ \ldots,\ \hat{t}_n$$

of $t_1,\ t_2,\ \ldots,\ t_n$, then these values can be naturally used to specify restrictions of the form

$$T_k^- \le t_k \le T_k^+, \quad k = 1, 2, \ldots, n, \tag{6.8}$$

by the proper choice of the bounds $T_k^-,\ T_k^+,\ k = 1, 2, \ldots, n$. Probably, the simplest choice is to set

$$T_k^- := \max\left\{a, \hat{t}_k - \frac{b-a}{n+1}\right\},$$

$$T_k^+ := \min\left\{\hat{t}_k + \frac{b-a}{n+1}, b\right\}$$

for $k = 1, 2, \ldots, n$. However, finer estimates may be available in specific situations. Knowing estimates of the form (6.8), instead of (4.3), we can check the relations

$$\varrho^{(k)} \ge \frac{T_k^+ - T_{k-1}^-}{4} \delta_{[T_{k-1}^-, T_k^+], \mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})}(\tilde{f}), \tag{6.9}$$

where $T_0^- = T_0^+ = a,\ T_{n+1}^- = T_{n+1}^+ = b$, and $\Omega^{(k)},\ k = 1, \ldots, n+1$, are suitably chosen sets satisfying (4.8). Similarly, instead of (4.6), we check the condition

$$r(K_k) < \frac{10}{3\big(T_k^+ - T_{k-1}^-\big)}, \tag{6.10}$$

where K_k is the Lipschitz matrix for the restriction of $\tilde{f}$ to $[t_{k-1}, t_k] \times \mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)}),\ k = 1, 2, \ldots, n+1$. Condition (6.10) is, clearly, preferable to (4.7).

Assuming (6.8), we formally make the problem more difficult because $t_1,\ t_2, \ldots, t_n$ must satisfy additional inequalities and cannot be arbitrary any longer. However, for a reasonable choice of the bounds based on the results of computations, inequalities (6.8), in fact, only state that we restrict ourselves to finding the unknown times in the regions where we have reasons to believe that they are present.

7. An Illustrative Example

We illustrate the approach described above by a model example of three-dimensional system

$$\begin{aligned} u_1'(t) &= u_2(t)u_3(t) - t^2 + \frac{67}{10}t - \frac{387}{100}, \\ u_2'(t) &= |u_3(t)|\, u_2(t) + q_2(t), \\ u_3'(t) &= |u_1(t)| + q_3(t) \end{aligned} \tag{7.1}$$

for $t \in [0, 1]$ with

$$q_2(t) := \begin{cases} t^2 - \frac{6}{5}t + \frac{33}{25} & \text{if} \quad t \in [0, t_3], \\ -t^2 + \frac{6}{5}t + \frac{17}{25} & \text{if} \quad t \in [t_3, 1], \end{cases} \tag{7.2}$$

and

$$q_3(t) := \begin{cases} -\dfrac{11}{4}t^2 + \dfrac{71}{20}t + \dfrac{2}{5} & \text{if} \quad t \in [0, t_1], \\[2ex] \dfrac{11}{4}t^2 - \dfrac{71}{20}t + \dfrac{8}{5} & \text{if} \quad t \in [t_1, 1], \end{cases} \tag{7.3}$$

where t_1 and t_3, $t_1 < t_3$, are unknown points from the interval $(0, 1)$. System (7.1) is considered with two-point nonlinear boundary conditions

$$u_1^2(0) - u_2^2(1) = 0, \qquad u_2(0)\, u_3(1) = -\frac{2}{25}, \qquad u_1(0) - u_3(1) = \frac{2}{5}. \tag{7.4}$$

We now pose the problem of finding the $\big[(1, -1; t_1), (-1, 1; t_2), (-1, 1; t_3)\big]$ solutions of (7.1), (7.4), where t_2 is a point lying between t_1 and t_3. It is necessary to determined the values of times t_1, t_2, and t_3, where the corresponding components of u change their signs.

It can be directly verified by computations that, for $t_1 = 1/5$, $t_2 = 2/5$, $t_3 = 4/5$, the function $u^* = (u_i^*)_{i=1}^3$ with the components

$$u_1^*(t) = \frac{11}{4}t^2 - \frac{71}{20}t + \frac{3}{5}, \qquad u_2^*(t) = t - \frac{2}{5}, \qquad u_3^*(t) = t - \frac{4}{5} \tag{7.5}$$

is a solution of the boundary-value problem (7.1), (7.4). It is easy to see that this solution has the $\big[(1, -1; 1/5),$ $(-1, 1; 2/5), (-1, 1; 4/5)\big]$ type in a sense of Definition 2.2.

We now use the approach described above. It is clear that (7.1) is a special case of (1.1) with $a = 0$, $b = 1$, $n = 3$, and f of the form

$$f(t, x_1, x_2, x_3, y_1, y_2, y_3) := \begin{pmatrix} (x_2 - y_2)(x_3 - y_3) - t^2 + \dfrac{67}{10}t - \dfrac{387}{100} \\[2ex] (x_3 + y_3)(x_2 - y_2) + q_2(t) \\[2ex] x_1 + y_2 + q_3(t) \end{pmatrix}, \tag{7.6}$$

and, hence, the previous argument is applicable. This explicit form (7.6) of f is, however, not necessary to write the corresponding system (2.4) because the function

$$\tilde{f} = (\tilde{f}_i)_{i=1}^3$$

specifying (2.4) can be constructed as in Remark 2.1 by using matrices (2.11):

$$\begin{aligned} M_1 &= \operatorname{diag}(\sigma_{10}, \sigma_{20}, \sigma_{30}), \qquad & M_2 &= \operatorname{diag}(\sigma_{11}, \sigma_{20}, \sigma_{30}), \\ M_3 &= \operatorname{diag}(\sigma_{11}, \sigma_{21}, \sigma_{30}), \qquad & M_4 &= \operatorname{diag}(\sigma_{11}, \sigma_{21}, \sigma_{31}). \end{aligned} \tag{7.7}$$

Since, in our case, we have $\sigma_{10} = 1$, $\sigma_{11} = -1$, $\sigma_{20} = -1$, $\sigma_{21} = 1$, $\sigma_{30} = -1$, $\sigma_{31} = 1$, equalities (7.7) imply that

$$M_1 = \operatorname{diag}(1, -1, -1), \qquad M_2 = \operatorname{diag}(-1, -1, -1),$$

$$M_3 = \operatorname{diag}(-1, 1, -1), \qquad M_4 = \operatorname{diag}(-1, 1, 1).$$

Then $|u_i|,\ i=1,3,$ on the kth interval $[t_{k-1},t_k],\ 1\le k\le 4,$ where $t_0=0$ and $t_4=1,$ should be replaced by the ith component of $M_k u$. In this way, we obtain

$$\tilde f_1(t,u_1,u_2,u_3)=u_2u_3-t^2+\frac{67}{10}t-\frac{387}{100} \tag{7.8}$$

for all $t\in[0,1],$ while $\tilde f_2$ and $\tilde f_3$ on the relevant subintervals are defined as follows:

$$\begin{aligned}\tilde f_2(t,u_1,u_2,u_3)&=-u_2u_3+t^2-\frac{6}{5}t+\frac{33}{25},\\ \tilde f_3(t,u_1,u_2,u_3)&=u_1-\frac{11}{4}t^2+\frac{71}{20}t+\frac{2}{5}\end{aligned} \tag{7.9}$$

for $t\in[0,t_1],$

$$\begin{aligned}\tilde f_2(t,u_1,u_2,u_3)&=-u_2u_3+t^2-\frac{6}{5}t+\frac{33}{25},\\ \tilde f_3(t,u_1,u_2,u_3)&=-u_1+\frac{11}{4}t^2-\frac{71}{20}t+\frac{8}{5}\end{aligned} \tag{7.10}$$

for $t\in[t_1,t_3]$ (these equations have the same form on $[t_1,t_2]$ and $[t_2,t_3]$), and

$$\begin{aligned}\tilde f_2(t,u_1,u_2,u_3)&=u_2u_3-t^2+\frac{6}{5}t+\frac{17}{25},\\ \tilde f_3(t,u_1,u_2,u_3)&=-u_1+\frac{11}{4}t^2-\frac{71}{20}t+\frac{8}{5}\end{aligned} \tag{7.11}$$

for $t\in[t_3,1]$. Hence, system (2.4) corresponding to (7.1) has the form

$$u_i'(t)=\tilde f_i(t,u_1(t),u_2(t),u_3(t)),\qquad i=1,2,3,\quad t\in[t_{k-1},t_k],\quad \le 1\le k\le 4, \tag{7.12}$$

with $(\tilde f_i)_{i=1}^3$ given by the respective equalities (7.8)–(7.11), and we pass from (7.1), (7.4) to problem (7.12), (7.4).

In order to apply the procedures described above, it is necessary to choose suitable domains and check the conditions. We now choose the sets $\Omega_0,\ \Omega_1,\dots,\Omega_4$ in (3.4) as follows:

$$\begin{aligned}\Omega_0&=\big\{(u_1,u_2,u_3)\colon 0.5\le u_1\le 0.7,\ -0.6\le u_2\le -0.3,\ -0.95\le u_3\le -0.6\big\},\\ \Omega_1&=\big\{(u_1,u_2,u_3)\colon -0.1\le u_1\le 0.1,\ -0.3\le u_2\le -0.1,\ -0.75\le u_3\le -0.5\big\},\\ \Omega_2&=\big\{(u_1,u_2,u_3)\colon -0.5\le u_1\le -0.25,\ -0.1\le u_2\le 0.1,\ -0.5\le u_3\le -0.3\big\},\\ \Omega_3&=\big\{(u_1,u_2,u_3)\colon -0.55\le u_1\le -0.3,\ 0.3\le u_2\le 0.5,\ -0.1\le u_3\le 0.1\big\},\\ \Omega_4&=\big\{(u_1,u_2,u_3)\colon -0.3\le u_1\le -0.1,\ 0.5\le u_2\le 0.7,\ 0.1\le u_3\le 0.3\big\}.\end{aligned} \tag{7.13}$$

Table 1

Exact Values of the Parameters of Solution (7.5) and Their Computed Approximations

u^*		$m = 0$	$m = 1$	$m = 2$	$m = 3$
$z_1^{(0)}$	0.6	0.5987479750	0.5999603161	0.6000012161	0.5999999745
$z_2^{(0)}$	–0.4	–0.4025198245	–0.4000793836	–0.3999975678	–0.4000000510
$z_3^{(0)}$	–0.8	–0.7704273551	–0.8000737079	–0.7999810861	–0.8000000219
$z_2^{(1)}$	–0.2	–0.2020709807	–0.2001071216	–0.199996485	–0.2000000620
$z_3^{(1)}$	–0.6	–0.5684050181	–0.6000878738	–0.5999800574	–0.6000000464
$z_1^{(2)}$	–0.38	–0.3838828957	–0.3801569730	–0.3799948547	–0.3800000942
$z_3^{(2)}$	–0.4	–0.3700284637	–0.3999682608	–0.3999842474	–0.3999999656
$z_1^{(3)}$	–0.48	–0.4842219409	–0.4800381441	–0.4799998221	–0.4800000226
$z_2^{(3)}$	0.4	0.3967271760	0.4000262560	0.3999978194	0.4000000159
$z_1^{(4)}$	–0.2	–0.2025151122	–0.2000954038	–0.1999968834	–0.2000000586
$z_2^{(4)}$	0.6	0.5987479750	0.5999603161	0.6000012161	0.5999999745
$z_3^{(4)}$	0.2	0.1987479750	0.1999603161	0.2000012161	0.1999999745
t_1	0.2	0.1988295851	0.1999917615	0.2000000919	0.1999999900
t_2	0.4	0.4003180698	0.4001083934	0.3999957228	0.4000000506
t_3	0.8	0.7979815572	0.8000579299	0.7999969454	0.8000000377

This choice is motivated by the fact that the zero-order approximate determining system (i.e., (6.2), (6.3) with $m = 0$) has roots lying in these sets; see the second column in Table 1. In Figs. 1a–c, we illustrate the plots of the zero-order approximation $U_0 = (U_{0i})_{i=1}^3$. Recall that, in order to obtain this approximation, we use solely functions (5.1), and no iterations have been carried out yet. We see that this piecewise linear function provides quite reasonable approximate values of the parameters (in particular, of the times t_1, t_2, and t_3). In general, the quality of approximation by U_0 increases with the number of equations (equal to the number of intermediate nodes).

Given sets (7.13), it is necessary to verify the conditions of Section 4 on the corresponding sets $\Omega_{0,1}, \ldots, \Omega_{3,4}$ defined according to (3.6). For this purpose, we use Remark 4.3 and choose suitable parallelepipeds $\Omega^{(k)} \supset \Omega_{k-1,k}$, $k = 1, \ldots, 4$:

$$\begin{aligned}
\Omega^{(1)} &:= \big\{(u_1, u_2, u_3)\colon -0.1 \le u_1 \le 0.7,\ -0.6 \le u_2 \le -0.1,\ -0.95 \le u_3 \le -0.5\big\},\\
\Omega^{(2)} &:= \big\{(u_1, u_2, u_3)\colon -0.5 \le u_1 \le 0.1,\ -0.3 \le u_2 \le 0.1,\ -0.75 \le u_3 \le -0.3\big\},\\
\Omega^{(3)} &:= \big\{(u_1, u_2, u_3)\colon -0.55 \le u_1 \le -0.25,\ -0.1 \le u_2 \le 0.5,\ -0.5 \le u_3 \le 0.1\big\},\\
\Omega^{(4)} &:= \big\{(u_1, u_2, u_3)\colon -0.55 \le u_1 \le -0.1,\ 0.3 \le u_2 \le 0.7,\ -0.1 \le u_3 \le 0.3\big\}.
\end{aligned} \tag{7.14}$$

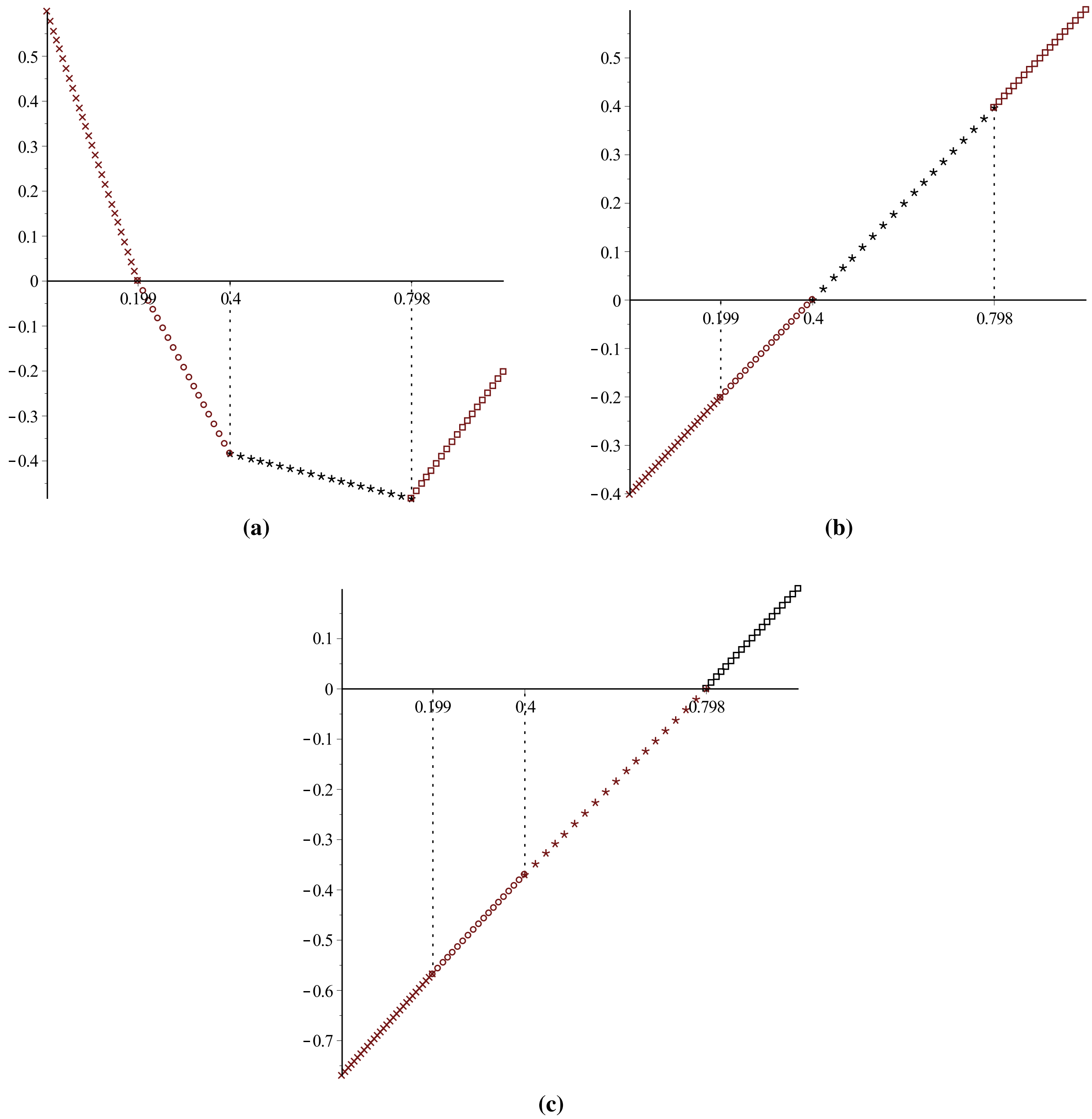

Fig. 1. Zero-order approximation: (a) first component, (b) second component, (c) third component.

We now verify conditions (4.9) on the sets

$$\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)}), \quad k = 1, \ldots, 4.$$

To do this, it is necessary to choose the vectors $\varrho^{(1)}, \ldots, \varrho^{(4)}$. Thus, e.g., we can set

$$\begin{gathered} \varrho^{(1)} = \mathrm{col}(0.2, 0.2, 0.2), \qquad \varrho^{(2)} = \varrho^{(1)}, \\ \varrho^{(3)} = \mathrm{col}(0.6, 0.2, 0.3), \qquad \varrho^{(4)} = \mathrm{col}(0.3, 0.2, 0.2). \end{gathered} \tag{7.15}$$

Hence, according to (3.8), it follows from (7.14) that

$$\begin{aligned}
\mathcal{O}_{\varrho^{(1)}}(\Omega^{(1)}) &= \big\{(u_1,u_2,u_3)\colon -0.3 \le u_1 \le 0.9,\ -0.8 \le u_2 \le 0.1,\ -1.15 \le u_3 \le -0.3\big\},\\
\mathcal{O}_{\varrho^{(2)}}(\Omega^{(2)}) &= \big\{(u_1,u_2,u_3)\colon -0.7 \le u_1 \le 0.3,\ -0.5 \le u_2 \le 0.3,\ -0.95 \le u_3 \le -0.1\big\},\\
\mathcal{O}_{\varrho^{(3)}}(\Omega^{(3)}) &= \big\{(u_1,u_2,u_3)\colon -1.15 \le u_1 \le 0.35,\ -0.3 \le u_2 \le 0.7,\ -0.8 \le u_3 \le 0.4\big\},\\
\mathcal{O}_{\varrho^{(4)}}(\Omega^{(4)}) &= \big\{(u_1,u_2,u_3)\colon -0.85 \le u_1 \le 0.2,\ 0.1 \le u_2 \le 0.9,\ -0.3 \le u_3 \le 0.5\big\}.
\end{aligned} \tag{7.16}$$

The direct computation performed by using (7.8)–(7.11) shows that the Lipschitz condition (4.5) for $\tilde{f}$ holds on $\mathcal{O}_{\varrho^{(1)}}(\Omega^{(1)}),\dots,\mathcal{O}_{\varrho^{(4)}}(\Omega^{(4)})$, respectively, with the matrices

$$\begin{aligned}
K_1 &= \begin{pmatrix} 0 & 1.15 & 0.8 \\ 0 & 1.15 & 0.8 \\ 1 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0.95 & 0.5 \\ 0 & 0.95 & 0.5 \\ 1 & 0 & 0 \end{pmatrix},\\
K_3 &= \begin{pmatrix} 0 & 0.8 & 0.7 \\ 0 & 0.8 & 0.7 \\ 1 & 0 & 0 \end{pmatrix}, \qquad K_4 = \begin{pmatrix} 0 & 0.5 & 0.9 \\ 0 & 0.5 & 0.9 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{7.17}$$

Thus, taking into account the rough approximations of $t_1,\ t_2,$ and t_3 obtained in the zero-order step (the second column of Table 1), we can assume, e.g., that the following bounds hold for regions (6.8), where it is necessary to find more precise values of these variables:

$$T_1^- \le t_1 \le T_1^+, \qquad T_2^- \le t_2 \le T_2^+, \qquad T_3^- \le t_3 \le T_3^+, \tag{7.18}$$

where

$$\begin{aligned}
T_1^- &:= 0.15, \qquad T_1^+ := 0.25, \qquad T_2^- := 0.35,\\
T_2^+ &:= 0.45, \qquad T_3^- := 0.75, \qquad T_3^+ := 0.85.
\end{aligned} \tag{7.19}$$

Under assumption (7.18), it follows from (7.17) that

$$r(K_1) \approx 1.6383 < \frac{40}{3} = \frac{10}{3T_1^+},$$

$$r(K_2) \approx 1.3268 < \frac{100}{9} = \frac{10}{3(T_2^+ - T_1^-)},$$

$$r(K_3) \approx 1.3274 < \frac{20}{3} = \frac{10}{3(T_3^+ - T_2^-)},$$

$$r(K_4) \approx 1.2311 < \frac{40}{3} = \frac{10}{3(1 - T_3^-)},$$

Table 2

Meaning of the Parameters in the Analyzed Example

$z_1^{(0)}$	$z_2^{(0)}$	$z_3^{(0)}$	$z_2^{(1)}$	$z_3^{(1)}$	$z_1^{(2)}$	$z_3^{(2)}$	$z_1^{(3)}$	$z_2^{(3)}$	$z_1^{(4)}$	$z_2^{(4)}$	$z_3^{(4)}$
$u_1(0)$	$u_2(0)$	$u_3(0)$	$u_2(t_1)$	$u_3(t_1)$	$u_1(t_2)$	$u_3(t_2)$	$u_1(t_3)$	$u_2(t_3)$	$u_1(1)$	$u_2(1)$	$u_3(1)$

which means that conditions (6.10) are satisfied. Furthermore, in view of (4.4), we get

$$\varrho^{(1)} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix} > \frac{T_1^+}{4} \delta_{[0,T_1^+],\mathcal{O}_{\varrho^{(1)}}(\Omega^{(1)})}(\tilde{f}) \approx \frac{0.25}{4} \begin{pmatrix} 2.6475 \\ 1.2725 \\ 1.9156 \end{pmatrix} \approx \begin{pmatrix} 0.1655 \\ 0.0795 \\ 0.1197 \end{pmatrix},$$

$$\varrho^{(2)} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix} > \frac{T_2^+ - T_1^-}{4} \delta_{[T_1^-,T_2^+],\mathcal{O}_{\varrho^{(2)}}(\Omega^{(2)})}(\tilde{f}) = \frac{0.3}{4} \begin{pmatrix} 2.59 \\ 0.94 \\ 1.57 \end{pmatrix} = \begin{pmatrix} 0.19425 \\ 0.0705 \\ 0.11775 \end{pmatrix},$$

$$\varrho^{(3)} = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.3 \end{pmatrix} > \frac{T_3^+ - T_2^-}{4} \delta_{[T_2^-,T_3^+],\mathcal{O}_{\varrho^{(3)}}(\Omega^{(3)})}(\tilde{f}) \approx \frac{0.5}{4} \begin{pmatrix} 3.59 \\ 0.9025 \\ 1.7401 \end{pmatrix} \approx \begin{pmatrix} 0.4488 \\ 0.1128 \\ 0.2175 \end{pmatrix},$$

$$\varrho^{(4)} = \begin{pmatrix} 0.3 \\ 0.2 \\ 0.2 \end{pmatrix} > \frac{1 - T_3^-}{4} \delta_{[T_3^-,1],\mathcal{O}_{\varrho^{(4)}}(\Omega^{(4)})}(\tilde{f}) \approx \frac{0.25}{4} \begin{pmatrix} 1.9575 \\ 0.8575 \\ 1.3656 \end{pmatrix} \approx \begin{pmatrix} 0.1223 \\ 0.0536 \\ 0.0854 \end{pmatrix}.$$

This means that conditions (6.9) are satisfied.

Thus, taking into account the observation made in Section 6, we conclude that the scheme based on Theorems 5.1–5.4 is applicable provided that inequalities (7.18) are true for $t_1,\ t_2$, and t_3. Note that, as shown by the numerical results, the true values of these variables indeed satisfy estimates (7.18). This situation is generic: when using this kind of computational schemes, it is always natural to choose the sets in conditions *after* getting some knowledge of where we are going to find the values of unknowns in the course of computation.

The scheme is now implemented as follows: We use equalities (5.1), (5.2) to construct the corresponding functions

$$u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)\colon [t_{k-1}, t_k] \to \mathbb{R}^3, \qquad 1 \le k \le 4, \quad m \ge 0. \tag{7.20}$$

These functions depend on the 12 scalar parameters listed in Table 2 and on the unknown times $t_1,\ t_2$, and t_3. By Theorem 5.1, functions (7.20) form convergent sequences as $m \to \infty$.

Note that, according to Section 5, the function $u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ is an approximation to the solution of the kth auxiliary two-point problem (3.2), (3.3) on the respective subintervals $[t_{k-1}, t_k],\ 1 \le k \le 4$. For this example, system (3.2), (3.3) means the following four problems:

Equation (7.12) on $[0, t_1]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2$ and $\tilde{f}_3$ from (7.9) under the conditions

$$\begin{aligned} &u_1(0) = z_1^{(0)}, \qquad u_2(0) = z_2^{(0)}, \qquad u_3(0) = z_3^{(0)}, \\ &u_1(t_1) = 0, \qquad u_2(t_1) = z_2^{(1)}, \qquad u_3(t_1) = z_3^{(1)}; \end{aligned} \tag{7.21}$$

Equation (7.12) on $[t_1, t_2]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2$ and $\tilde{f}_3$ from (7.10) under the conditions

$$\begin{aligned} &u_1(t_1) = 0, \qquad u_2(t_1) = z_2^{(1)}, \qquad u_3(t_1) = z_3^{(1)}, \\ &u_1(t_2) = z_1^{(2)}, \qquad u_2(t_2) = 0, \qquad u_3(t_2) = z_3^{(2)}; \end{aligned} \tag{7.22}$$

Equation (7.12) on $[t_2, t_3]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2$ and $\tilde{f}_3$ from (7.10) under the conditions

$$\begin{aligned} &u_1(t_2) = z_1^{(2)}, \qquad u_2(t_2) = 0, \qquad u_3(t_2) = z_3^{(2)}, \\ &u_1(t_3) = z_1^{(3)}, \qquad u_2(t_3) = z_2^{(3)}, \qquad u_3(t_3) = 0; \end{aligned} \tag{7.23}$$

Equation (7.12) on $[t_3, 1]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2$ and $\tilde{f}_3$ from (7.11) under the conditions

$$\begin{aligned} &u_1(t_3) = z_1^{(3)}, \qquad u_2(t_3) = z_2^{(3)}, \qquad u_3(t_3) = 0, \\ &u_1(1) = z_1^{(4)}, \qquad u_2(1) = z_2^{(4)}, \qquad u_3(1) = z_3^{(4)}. \end{aligned} \tag{7.24}$$

However, the auxiliary problems (7.21)–(7.24) are not directly treated in the course of computations, which involves only functions (7.20). The approximate solutions of the given problem (7.1), (7.4) are constructed, on the respective subintervals, in the form

$$U_m(t) := u_m^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big), \qquad t \in [t_{k-1}, t_k], \quad k = 1, \dots, 4,$$

where m is fixed and $z^{(j)}$, $j = 0, \dots, 4$, are vectors of the form (5.19) satisfying the mth approximate determining system (6.2), (6.3):

$$z^{(k)} - z^{(k-1)} - \int_{t_{k-1}}^{t_k} \tilde{f}\big(s, u_m^{(k)}(s, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) ds = 0, \quad k = 1, 2, \dots, 4,$$

$$\begin{aligned} &\big(u_1^{(1)}(0, z^{(0)}, z^{(1)}, 0, t_0)\big)^2 - \big(u_2^{(4)}(1, z^{(3)}, z^{(4)}, t_3, 1)\big)^2 = 0, \\ &u_2^{(1)}(0, z^{(0)}, z^{(1)}, 0, t_0)\, u_3^{(4)}(1, z^{(3)}, z^{(4)}, t_3, 1) = -\frac{2}{25}, \\ &u_1^{(1)}(0, z^{(0)}, z^{(1)}, 0, t_0) - u_3^{(4)}(1, z^{(3)}, z^{(4)}, t_3, 1) = \frac{2}{5}. \end{aligned} \tag{7.25}$$

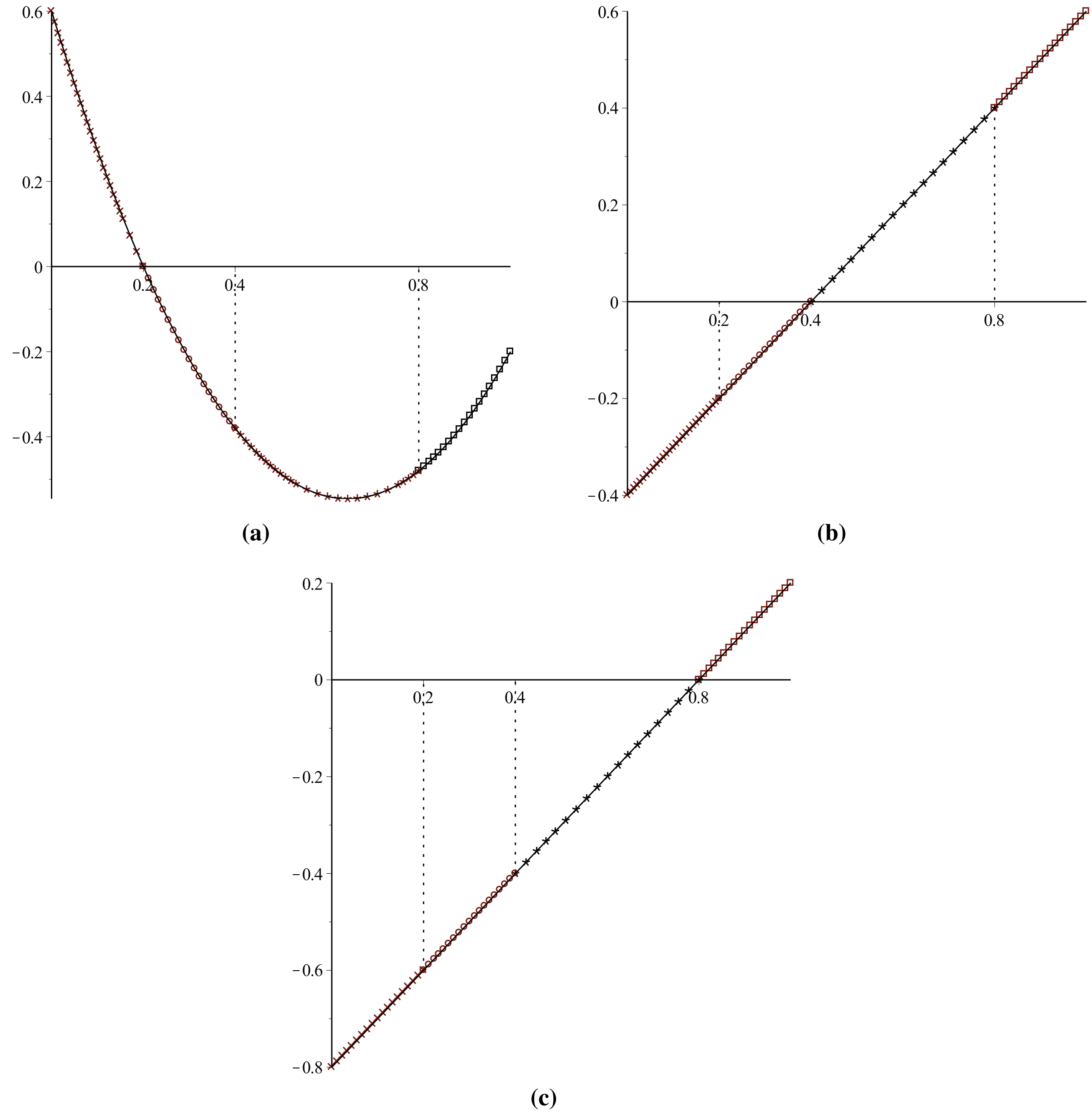

Fig. 2. Exact solution (solid line) and the third approximation: (a) first component, (b) second component, (c) third component.

In order to determine the values of the parameters in step m, equations (7.25) are numerically solved for $z^{(j)} \in \Omega_j,\ j = 0, \dots, 4$, and $t_i \in [T_i^-, T_i^+],\ i = 1, 2, 3$. An initial hint for the region where the roots should be sought is obtained by using the zero-order approximation $(m = 0)$, whose plots are shown in Figs. 1a–c. We have used the Maple-14 software to perform all necessary computations.

The numerical values of the 15 unknown parameters obtained from (7.25) for the first three steps of the iterative process are shown in Table 1. We see that the approximate values in the third iteration are very close to the exact values.

The plots of the respective components of the exact solution (7.5) and the approximate $\big[(1, -1; t_1), (-1, 1; t_2), (-1, 1; t_3)\big]$ solution $U_3 = (U_{3i})_{i=1}^3$ of problem (7.1), (7.4) corresponding to the numerical values from Table 1 are shown in Figs. 2a–c. The curves corresponding to the subintervals $[t_{k-1}, t_k],\ k = 1, \dots, 4$, with the values of $t_1,\ t_2$, and t_3 computed in the third step are plotted by using different symbols.

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Parametrisation for boundary value problems with transcendental non-linearities using polynomial interpolation

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. A constructive technique of analysis involving parametrisation and polynomial interpolation is suggested for general non-local problems for ordinary differential systems with locally Lipschitzian transcendental non-linearities. The practical application of the approach is shown on a numerical example.

Keywords: boundary value problem, transcendental non-linearity, parametrisation, successive approximations, Chebyshev nodes, interpolation

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1 Introduction

The present note deals with parametrisation techniques for constructive investigation of boundary value problems and its purpose is to provide a justification of the *polynomial* version of the method suggested in [14].

We consider the non-local boundary value problem

$$u'(t) = f(t, u(t)), \qquad t \in [a, b], \tag{1.1}$$

$$\phi(u) = \gamma, \tag{1.2}$$

where $\phi : C([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is a non-linear vector functional, $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in a certain bounded set, and $\gamma \in \mathbb{R}^n$ is a given vector.

By a solution of the problem (1.1), (1.2) we understand a continuously differentiable vector function with property (1.2) satisfying (1.1) everywhere on $[a, b]$.

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The idea of our approach (see, e. g., [14, 16, 17]) is based on the reduction (1.1), (1.2) to a family of simpler auxiliary problems with two-point linear separated conditions at a and b:

$$u(a) = \xi, \qquad u(b) = \eta, \tag{1.3}$$

where ξ and η are unknown parameters. By doing so, one can use in the non-local case the techniques adopted to two-point problems [14].

2 Notation and preliminary results

In order to use the reduction to two-point problems (1.1), (1.3), we need some results from [14]. The study of problems (1.1), (1.3) in [14] is based on properties of the iteration sequence $\{u_m(\cdot,\xi,\eta) : m \geq 0\}$ defined as follows:

$$u_0\left(t,\xi,\eta\right) := \left(1 - \frac{t-a}{b-a}\right)\xi + \frac{t-a}{b-a}\eta, \tag{2.1}$$

$$\begin{aligned} u_m\left(t,\xi,\eta\right) := {} & u_0(t,\xi,\eta) + \int_a^t f\left(s, u_{m-1}\left(s,\xi,\eta\right)\right) ds \\ & - \frac{t-a}{b-a}\int_a^b f\left(s, u_{m-1}\left(s,\xi,\eta\right)\right) ds, \qquad t \in [a,b],\ m = 1,2,\ldots \end{aligned} \tag{2.2}$$

Fix certain closed bounded sets D_0, D_1 in $\mathbb{R}^n$ and assume that we are looking for solutions u of problem (1.1), (1.3) with $u(a) \in D_0$ and $u(b) \in D_1$. Put

$$\Omega := \{(1-\theta)\xi + \theta\eta : \xi \in D_0,\ \eta \in D_1,\ \theta \in [0,1]\} \tag{2.3}$$

and, for any $\varrho \in \mathbb{R}^n_+$, define the set

$$\Omega_\varrho := O_\varrho(\Omega), \tag{2.4}$$

where $O_\varrho(\Omega) := \bigcup_{z\in\Omega} O_\varrho(z)$ and $O_\varrho(\xi) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \varrho\}$ for any ξ. Here and below, the operations $\leq$ and $|\cdot|$ are understood componentwise. Set (2.4) is a *componentwise ϱ-neighbourhood* of Ω.

Introduce some notation. Given a domain $D \subset \mathbb{R}^n$, we write $f \in \mathrm{Lip}_K(D)$ if K is an $n \times n$ matrix with non-negative entries and the inequality

$$|f(t,u) - f(t,v)| \leq K\,|u-v| \tag{2.5}$$

holds for all $\{u,v\} \subset D$ and $t \in [a,b]$. We also put

$$\delta_{[a,b],D}(f) := \sup_{(t,x)\in[a,b]\times D} f(t,x) - \inf_{(t,x)\in[a,b]\times D} f(t,x). \tag{2.6}$$

The computation of the greatest and least lower bounds for vector functions is understood in the componentwise sense.

The following statement is a combination of Proposition 1 and Theorem 3 from [14].

Theorem 2.1 ([14]). *Let there exist a non-negative vector ϱ satisfying the inequality*

$$\varrho \geq \frac{b-a}{4}\,\delta_{[a,b],\Omega_\varrho}(f). \tag{2.7}$$

Assume, furthermore, that there exists a non-negative matrix K such that

$$r(K) < \frac{10}{3(b-a)} \tag{2.8}$$

and $f \in \mathrm{Lip}_K(\Omega_\varrho)$. Then, for all fixed $(\xi,\eta) \in D_0 \times D_1$:

1. *For every m, the function $u_m(\cdot, \xi, \eta)$ satisfies the two-point separated boundary conditions* (1.3) *and*

$$\{u_m(t, \xi, \eta) : t \in [a, b]\} \subset \Omega_\varrho.$$

2. *The limit*

$$u_\infty(t, \xi, \eta) = \lim_{m\to\infty} u_m(t, \xi, \eta) \tag{2.9}$$

exists uniformly in $t \in [a, b]$. The function $u_\infty(\cdot, \xi, \eta)$ satisfies the two-point conditions (1.3).

3. *The function $u_\infty(\cdot, \xi, \eta)$ is a unique solution of the integral equation*

$$u(t) = \xi + \int_a^t f(s, u(s))ds - \frac{t-a}{b-a}\int_a^b f(s, u(s))ds + \frac{t-a}{b-a}(\eta - \xi), \qquad t \in [a, b], \tag{2.10}$$

or, equivalently, of the Cauchy problem

$$\begin{aligned} u'(t) &= f(t, u(t)) + \frac{1}{b-a}\Delta(\xi, \eta), \qquad t \in [a, b], \\ u(a) &= \xi, \end{aligned} \tag{2.11}$$

where $\Delta : D_0 \times D_1 \to \mathbb{R}^n$ is a mapping given by the formula

$$\Delta(\xi, \eta) := \eta - \xi - \int_a^b f(s, u_\infty(s, \xi, \eta))ds. \tag{2.12}$$

4. *The following error estimate holds:*

$$|u_\infty(t, \xi, \eta) - u_m(t, \xi, \eta)| \le \frac{10}{9}\alpha_1(t) K_*^m (1_n - K_*)^{-1} \delta_{[a,b],\Omega_\varrho}(f), \tag{2.13}$$

for any $t \in [a, b]$ and $m \ge 0$, where

$$K_* := \frac{3}{10}(b-a)K \tag{2.14}$$

and

$$\alpha_1(t) := 2(t-a)\left(1 - \frac{t-a}{b-a}\right), \qquad t \in [a, b]. \tag{2.15}$$

In (2.13) and everywhere below, the symbol 1_n stands for the unit matrix of dimension n.

Theorem 2.2 ([14, Proposition 8])**.** *Under the assumption of Theorem 2.1, the function $u_\infty(\cdot, \xi, \eta) : [a, b] \times D_0 \times D_1 \to \mathbb{R}^n$ defined by* (2.9) *is a solution of problem* (1.1), (1.2) *if and only if the pair of vectors (ξ, η) satisfies the system of $2n$ equations*

$$\Delta(\xi, \eta) = 0, \tag{2.16}$$

$$\phi(u_\infty(\cdot, \xi, \eta)) = \gamma, \tag{2.17}$$

where Δ is given by (2.12).

Equations (2.16), (2.17) are usually referred to as *determining* equations because their roots determine solutions of the original problem. This system, in fact, determines all possible solutions of the original boundary value problem the graphs of which are contained in the region under consideration.

Theorem 2.3 ([14, Theorem 9]). *Let $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with a certain ϱ satisfying (2.7) and K such that (2.8) holds. Then:*

1. if there exists a pair of vectors $(\xi, \eta) \in D_0 \times D_1$ satisfying (2.16), (2.17), then the non-local problem (1.1), (1.2) has a solution $u(\cdot)$ such that

$$\{u(t) : t \in [a, b]\} \subset \Omega_\varrho \tag{2.18}$$

and $u(a) = \xi$, $u(b) = \eta$;

2. if problem (1.1), (1.2) has a solution $u(\cdot)$ such that (2.18) holds and $u(a) \in D_0$, $u(b) \in D_1$, then the pair $(u(a), u(b))$ is a solution of system (2.16), (2.17).

The solvability of the determining system (2.16), (2.17) can be analysed by using its mth approximate version

$$\eta - \xi - \int_a^b f(s, u_m(s, \xi, \eta))ds = 0, \tag{2.19}$$

$$\phi(u_m(\cdot, \xi, \eta)) = \gamma, \tag{2.20}$$

where m is fixed, similarly to [10,12,13,15]. Equations (2.19), (2.20), in contrast to (2.16), (2.17), involve only terms which are obtained in a finite number of steps.

The explicit computation of functions (2.2) (and, as a consequence of this, the construction of equations (2.19), (2.20)) may be difficult or impossible if the expression for f involves complicated non-linearities with respect to the space variable, which causes problems with symbolic integration. In order to facilitate the computation of $u_m(\cdot, \xi, \eta)$, $m \geq 0$, one can use a *polynomial* version of the iterative scheme (2.2), in which the results of iteration are replaced by suitable interpolation polynomials before passing to the next step. This scheme is described below.

3 Some results from interpolation theory

Recall some results of the theory of approximations [2, 3, 6]. In a similar situation, we have used these facts in [11].

Denote by $\mathcal{P}_q$ a set of all polynomials of degree not higher than q, $q \geq 1$, on $[a, b]$. For any continuous function $y : [a, b] \to \mathbb{R}$, there exists a unique polynomial $p_q^* \in \mathcal{P}_q$, for which $\max_{t\in[a,b]} |y(t) - p_q^*(t)| = E_q(y)$, where

$$E_q(y) := \inf_{p\in\mathcal{P}_q} \max_{t\in[a,b]} |y(t) - p(t)|. \tag{3.1}$$

This p_q^* is the polynomial of the *best uniform approximation* of y in $\mathcal{P}_q$ and the number $E_q(y)$ is called the *error of the best uniform approximation*.

For a given continuous function $y :[a, b] \to \mathbb{R}$ and a natural number q, denote by $L_q y$ the Lagrange interpolation polynomial of degree q such that

$$(L_q y)(t_i) = y(t_i), \qquad i = 1, 2, \dots, q + 1, \tag{3.2}$$

where

$$t_i = \frac{b - a}{2} \cos \frac{(2i - 1)\pi}{2(q + 1)} + \frac{a + b}{2}, \qquad i = 1, 2, \dots, q + 1, \tag{3.3}$$

are the Chebyshev nodes translated from $(-1, 1)$ to the interval (a, b) (see, e. g., [7]).

Proposition 3.1 ([7, p. 18]). *For any $q \geq 1$ and a continuous function $y : [a, b] \to \mathbb{R}$, the corresponding interpolation polynomial (3.2) constructed with the Chebyshev nodes (3.3) admits the estimate*

$$\left|y(t) - (L_q y)(t)\right| \leq \left(\frac{2}{\pi}\ln q + 1\right) E_q(y), \qquad t \in [a, b]. \tag{3.4}$$

Recall that the *modulus of continuity* [5, p. 116] of a continuous function $y : [a, b] \to \mathbb{R}$ is the function $\delta \mapsto \omega(y; \delta)$, where

$$\omega(y; \delta) := \sup\{|y(t) - y(s)| : \{t, s\} \subset [a, b],\ |t - s| \leq \delta\} \tag{3.5}$$

for all positive δ. Note that $\omega(y; \cdot)$ is a continuous non-decreasing function on $(0, \infty)$. A function y is uniformly continuous if and only if $\lim_{\delta \to 0} \omega(y; \delta) = 0$ [5, p. 131].

Proposition 3.2 (Jackson's theorem; [6, p. 22]). *If $y \in C([a, b], \mathbb{R}), q \geq 1$, then*

$$E_q(y) \leq 6\,\omega\Big(y; \frac{b - a}{2q}\Big). \tag{3.6}$$

A function $y : [a, b] \to \mathbb{R}$ is said to satisfy the *Dini–Lipschitz condition* (see, e. g., [3, p. 50]) if its modulus of continuity has the property

$$\lim_{\delta \to 0} \omega(y; \delta) \ln \delta = 0.$$

It follows from (3.6) that

$$\lim_{q \to \infty} E_q(y) \ln q = 0 \tag{3.7}$$

for any y satisfying the Dini–Lipschitz condition. In view of (3.4), equality (3.7) ensures the uniform convergence of Lagrange interpolation polynomials at Chebyshev nodes for this class of functions. In particular, every α-Hölder continuous function $[a, b] \to \mathbb{R}$ with $\alpha > 0$ satisfies the Dini–Lipschitz condition.

4 Polynomial successive approximations

Rewrite (2.2) in the form

$$u_m(t, \xi, \eta) = u_0(t, \xi, \eta) + (\Lambda N_f u_{m-1}(\cdot, \xi, \eta)])(t), \qquad t \in [a, b],\ m = 1, 2, \ldots, \tag{4.1}$$

where Λ is the linear operator in the space of continuous functions defined by the formula

$$(\Lambda y)(t) := \int_a^t y(s) ds - \frac{t - a}{b - a} \int_a^b y(s) ds, \qquad t \in [a, b], \tag{4.2}$$

and N_f is the Nemytskii operator generated by the non-linearity from (1.1),

$$(N_f y)(t) := f(t, y(t)), \qquad t \in [a, b], \tag{4.3}$$

for any continuous $y : [a, b] \to \mathbb{R}^n$.

Fix a natural number q and extend the notation $L_q y$ to vector functions by putting

$$L_q y := \mathrm{col}(L_q y_1, L_q y_2, \ldots, L_q y_n) \tag{4.4}$$

for any continuous $y : [a,b] \to \mathbb{R}^n$. In (4.4), $L_q y_i$ is the qth degree interpolation polynomial for y_i at the Chebyshev nodes (3.3). By analogy to (4.4), put

$$E_q y = \operatorname{col}(E_q y_1, E_q y_2, \dots, E_q y_n). \tag{4.5}$$

If $D \subset \mathbb{R}^n$ is a closed domain and $f : [a,b] \times D \to \mathbb{R}^n$, put

$$l_{q,D}(f) := \left(\frac{2}{\pi}\ln q + 1\right) \sup_{p \in \mathcal{P}_{q+1,D}} E_q(N_f p), \tag{4.6}$$

where

$$\mathcal{P}_{q,D} := \{u : u \in \mathcal{P}_q^n,\ u([a,b]) \subset D\} \tag{4.7}$$

with $\mathcal{P}_q^n := \mathcal{P}_q \times \cdots \times \mathcal{P}_q$. The second multiplier in (4.6) is the least upper bound of errors of best uniform approximations of the functions obtained by substitution into the right-hand side of equation (1.1) of vector polynomials of degree $\le q+1$ with values in D.

Introduce now a modified iteration process keeping formula (2.1) for $u_0(\cdot,\xi,\eta)$:

$$v_0^q(\cdot,\xi,\eta) := u_0(\cdot,\xi,\eta) \tag{4.8}$$

and replacing (4.1) by the formula

$$v_m^q(t,\xi,\eta) := u_0(t,\xi,\eta) + (\Lambda L_q N_f v_{m-1}^q(\cdot,\xi,\eta))(t), \qquad t \in [a,b],\ m = 1,2,\dots \tag{4.9}$$

For any $q \ge 1$, formula (4.9) defines a vector polynomial $v_m^q(\cdot,\xi,\eta)$ of degree $\le q+1$ (in particular, all these functions are continuously differentiable), which, moreover, satisfies the two-point boundary conditions (1.3). The coefficients of the interpolation polynomials depend on the parameters ξ and η.

Similarly to (4.1), functions (4.9) can also be used to study the auxiliary problems (1.1), (1.3).

Let H_k^β, where $k \in \mathbb{R}_+^n$, $k_i \ge 0$, $0 < \beta_i \le 1$, $i = 1,2,\dots,n$, be the set of vector functions $y : [a,b] \to \mathbb{R}^n$ satisfying the Hölder conditions

$$|y_i(t) - y_i(s)| \le k_i |t-s|^{\beta_i} \tag{4.10}$$

for all $\{t,s\} \subset [a,b]$, $i = 1,2,\dots,n$. Now we can state the "polynomial" version of Theorem 2.1.

Theorem 4.1. *Let there exist a non-negative vector ϱ such that*

$$\varrho \ge \frac{b-a}{4}\left(\delta_{[a,b],\Omega_\varrho}(f) + 2 l_{q,\Omega_\varrho}(f)\right) \tag{4.11}$$

and $f \in \operatorname{Lip}_K(\Omega_\varrho)$ with a certain matrix K satisfying (2.8). Furthermore, let there exist vectors c and β with $c_i \ge 0$, $0 < \beta_i \le 1$, $i = 1,2,\dots,n$, such that

$$f(\cdot,\xi) \in H_c^\beta \tag{4.12}$$

for all fixed $\xi \in \Omega_\varrho$. Then, for all fixed $(\xi,\eta) \in D_0 \times D_1$:

1. For any $m \ge 0$, $q \ge 1$, the function $v_m^q(\cdot,\xi,\eta)$ is a vector polynomial of degree $q+1$ having values in Ω_ϱ and satisfying the two-point conditions (1.3).

2. *The limits*

$$v^q_\infty(\cdot,\xi,\eta) := \lim_{m\to\infty} v^q_m(\cdot,\xi,\eta), \qquad v_\infty(\cdot,\xi,\eta) := \lim_{q\to\infty} v^q_\infty(\cdot,\xi,\eta) \tag{4.13}$$

exist uniformly on $[a,b]$. *Functions* (4.13) *satisfy conditions* (1.3).

3. *The estimate*

$$\left|u_\infty(t,\xi,\eta) - v^q_m(\cdot,\xi,\eta)\right| \le \frac{10}{9}\alpha_1(t)K^m_*(1_n - K_*)^{-1}\left(\delta_{[a,b],\Omega_\varrho}(f) + l_{q,\Omega_\varrho}(f)\right) \tag{4.14}$$

holds for any $t \in [a,b]$, $m \ge 0$, $q \ge 1$, *where* K_* *and* α_1 *are given by* (2.14), (2.15).

The proof of this theorem is given in Section 5.1. Note that v_∞ coincides with u_∞ appearing in Theorems 2.1 and 2.2.

Similarly to (2.19), (2.20), in order to study the solvability of the determining system (2.16), (2.17), one can use its mth approximate polynomial version

$$\eta - \xi = \int_a^b (L_q N_f v^q_m(\cdot,\xi,\eta))(s)ds, \tag{4.15}$$

$$\phi(v^q_m(\cdot,\xi,\eta)) = \gamma, \tag{4.16}$$

which can be regarded as an approximate version of (2.19), (2.20). If $(\hat\xi,\hat\eta)$ is a root of (4.15), (4.16) in a particular region, then the function

$$U^q_m(t) := v^q_m\left(t,\hat\xi,\hat\eta\right), \qquad t\in[a,b], \tag{4.17}$$

provides the mth *polynomial approximation* to a solution of the original problem with the corresponding localisation of initial data. Of course, system (2.19), (2.20) may have multiple roots; in such cases, these roots determine different solutions.

It should be noted that, under conditions of Theorem 4.1, the function $N_f v^q_{m-1}(\cdot,\xi,\eta))$ appearing in (4.9) always satisfies the Dini–Lipschitz condition and, therefore, the corresponding interpolation polynomials at Chebyshev nodes uniformly converge to it as q grows to ∞. This follows from Lemma 5.1 of the next section.

Condition (4.11) on ϱ assumed in Theorem 4.1 is stronger than (2.7) of Theorem 2.1 due to the presence of an additional positive term on the right-hand side. A stronger version of (2.7) is needed in order to ensure that the values of iterations do not escape from the set where the Lipschitz condition on f is assumed, for which purpose (2.7) is sufficient in the case of iterations (2.1), (2.2).

The value $E_q(N_f p)$, where $p \in \mathcal{P}_{q+1}$, appearing in (4.6) essentially depends on the character of the non-linearity f. In particular, if f is linear, then $E_q(N_f p)$ is the error of the best uniform approximation of a polynomial of degree $\le q+1$ by polynomials of degree $\le q$.

In spite of the presence of an additional expression in (4.11), for which the theorem does not provide explicit estimates, one may however say that, technically, it is (2.7) that plays the most important role here because the extra term is due to the polynomial approximation, the quality of which grows with q. One can treat this in a different way as follows. Instead of assuming condition (4.11), let us suppose that there exists a non-negative vector ϱ such that

$$\varrho \ge \frac{b-a}{4}\left(\delta_{[a,b],\Omega_\varrho}(f) + r\right) \tag{4.18}$$

with a certain strictly positive vector r. Put

$$w_0(\cdot,\xi,\eta) := u_0(\cdot,\xi,\eta), \tag{4.19}$$
$$w_m(t,\xi,\eta) := u_0(t,\xi,\eta) + (\Lambda L_{q_m} N_f w_{m-1}(\cdot,\xi,\eta))(t), \qquad t \in [a,b],\ m = 1,2,\dots \tag{4.20}$$

where $\{q_m : m \geq 1\} \subset \mathbb{N}$; the choice of this sequence will be discussed below. The condition (2.8) on the maximal in modulus eigenvalue of the Lipschitz matrix K for f in (1.1) is left intact.

Repeating almost word for word the argument from the proof of Theorem 4.1 (see Section 5.1), we find that the sequence $\{w_m(\cdot,\xi,\eta) : m \geq 0\}$ defined according to (4.19), (4.20) converges to the same limit as $\{u_m(\cdot,\xi,\eta) : m \geq 0\}$ given by (4.1) provided that

$$\sup_{\xi \in D_0,\ \eta \in D_1} \sup_{m \geq 1} \Big(\frac{2}{\pi} \ln q_m + 1\Big) E_{q_m}(N_f w_{m-1}^{q_m}(\cdot,\xi,\eta)) \leq \frac{1}{2} r, \tag{4.21}$$

where r is the vector appearing in (4.18). Although (4.21) involves the members of sequence (4.19), (4.20), other assumptions on f (namely, (4.12) and the Lipschitz condition in the space variable) and Jackson's theorem (Proposition 3.2) guarantee that, given any value of r in (4.18), the corresponding condition (4.21) can always be satisfied by choosing $q_1, q_2, \dots$ appropriately. This means that the following is true.

Proposition 4.2. *Under conditions (2.8), (4.12), and (4.18), sequence (4.19), (4.20) uniformly converges provided that q_m is chosen large enough at every step m.*

In that case, sequence (4.19), (4.20) will serve the same purpose as sequence (4.8), (4.9) under the assumptions of Theorem 4.1.

The argument above relies on the knowledge of smallness of the related term appearing on the left-hand side of (4.21). It is however natural to expect that such quantities should diminish if the number of nodes gets larger. To see this, let us now assume conditions somewhat stronger than those of Theorem 4.1.

Assume that, instead of (2.8), the matrix K appearing in the inclusion $f \in \mathrm{Lip}_K(\Omega_\varrho)$ satisfies the condition

$$r(K) < \frac{2}{b-a}. \tag{4.22}$$

Theorem 4.3. *Let there exist a non-negative vector ϱ and positive vector r such that (4.18) holds and $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with K satisfying (4.22). Assume that $f(\cdot,\xi)$ is Lipschitzian with some constant vector c for all fixed $\xi \in \Omega_\varrho$. Then the iteration process (4.19), (4.20) can be made convergent by choosing $q_m = q$, $r = 1,2,\dots$, with q sufficiently large.*

In other words, under conditions of Theorem 4.3, the iteration process (4.19), (4.20) reduces to (4.8), (4.9) with q large enough.

5 Proofs

5.1 Proof of Theorem 4.1

We shall use several auxiliary statements formulated below.

Lemma 5.1. *Let $D \subset \mathbb{R}^n$ and $f : [a,b] \times D \to \mathbb{R}^n$ be a function satisfying condition (4.12) on D with certain vectors c and $\beta = (\beta_i)_{i=1}^n$, $0 < \beta_i \le 1$, $i = 1, 2, \ldots, n$. Let $f \in \mathrm{Lip}_K(D)$ with a certain $n \times n$ matrix K with non-negative entries. If $u \in H_{\tilde{c}}^{\tilde{\beta}}$ with $\tilde{\beta} = (\tilde{\beta}_i)_{i=1}^n$, $0 < \tilde{\beta}_i \le 1$, $i = 1, 2, \ldots, n$, then*

$$N_f u \in H^{\mu}_{K\tilde{c}+c}, \tag{5.1}$$

where $\mu := \min\{\beta, \tilde{\beta}\}$.

Proof. Assume that $u \in H_{\tilde{c}}^{\tilde{\beta}}$ and the values of u lie in D. For the sake of brevity, introduce the notation $t^{\beta} := \mathrm{col}(t^{\beta_1}, t^{\beta_2}, \ldots, t^{\beta_n})$ for any $t \in [a,b]$. Using (4.12) and the Lipschitz condition for f, we obtain

$$\begin{aligned} \left|(N_f u)(t) - (N_f u)(s)\right| &= |f(t, u(t)) - f(t, u(s)) + f(t, u(s)) - f(s, u(s))| \\ &\le K\,|u(t) - u(s)| + c\,|t - s|^{\beta} \\ &\le K\tilde{c}\,|t - s|^{\tilde{\beta}} + c\,|t - s|^{\beta} \\ &\le (K\tilde{c} + c)\,|t - s|^{\mu} \end{aligned}$$

with $\mu = \min\{\beta, \tilde{\beta}\}$, i.e., the function $N_f u$ satisfies a condition of form (4.10), which proves relation (5.1). □

Let the functions $\alpha_m : [a,b] \to \mathbb{R}_+$, $m \ge 0$, be defined by the recurrence relation

$$\alpha_0(t) := 1, \tag{5.2}$$

$$\alpha_{m+1}(t) := \left(1 - \frac{t-a}{b-a}\right) \int_a^t \alpha_m(s)ds + \frac{t-a}{b-a} \int_t^b \alpha_m(s)ds, \qquad m = 0, 1, 2, \ldots \tag{5.3}$$

For $m = 0$, formula (5.3) reduces to (2.15).

Lemma 5.2 ([8, Lemma 3])**.** *For any continuous function $y : [a,b] \to \mathbb{R}$, the estimate*

$$\left|\int_a^t \left(y(\tau) - \frac{1}{b-a}\int_a^b y(s)ds\right)d\tau\right| \le \frac{1}{2}\alpha_1(t)\left(\max_{s\in[a,b]} f(s) - \min_{s\in[a,b]} f(s)\right), \qquad t \in [a,b], \tag{5.4}$$

holds, where $\alpha_1(\cdot)$ is given by (2.15).

Lemma 5.3 ([9, Lemma 3.16])**.** *The following estimates hold for all $t \in [a,b]$:*

$$\begin{aligned} \alpha_{m+1}(t) &\le \frac{10}{9}\left(\frac{3(b-a)}{10}\right)^m \alpha_1(t), \qquad m \ge 0, \\ \alpha_{m+1}(t) &\le \frac{3}{10}(b-a)\,\alpha_m(t), \qquad m \ge 2. \end{aligned} \tag{5.5}$$

Let us now turn to the *proof* of Theorem 4.1. Fix $\xi \in D_0$, $\eta \in D_1$, $q \ge 1$, and put

$$y_m^q := N_f v_m^q\,(\cdot, \xi, \eta) \tag{5.6}$$

for $m \ge 0$. We need to show that

$$\{v_m^q(t, \xi, \eta) : t \in [a,b]\} \subset \Omega_{\varrho} \tag{5.7}$$

for any m. Obviously, (5.7) holds if $m = 0$.

For $m \geq 1$, in view of (2.6), (4.2) and (5.6), Lemma 5.2 yields the componentwise estimates

$$\begin{aligned}
|(\Lambda y_m^q)(t)| &\leq \frac{1}{2}\alpha_1(t)\Big(\max_{s\in[a,b]} y_m^q(s) - \min_{s\in[a,b]} y_m^q(s)\Big) \\
&= \frac{1}{2}\alpha_1(t)\Big(\max_{s\in[a,b]} f(s, v_m^q(s,\xi,\eta)) - \min_{s\in[a,b]} f(s, v_m^q(\cdot,\xi,\eta))\Big) \\
&\leq \frac{1}{2}\alpha_1(t)\delta_{[a,b],\Omega_\varrho}(f) \\
&\leq \frac{1}{4}(b-a)\delta_{[a,b],\Omega_\varrho}(f) \qquad (5.8)
\end{aligned}$$

for all $t \in [a,b]$. In (5.8), we have used the equality

$$\max_{t\in[a,b]} \alpha_1(t) = \frac{1}{2}(b-a) \qquad (5.9)$$

which follows directly from (2.15). Furthermore, using relations (5.4), (5.9) and estimate (3.4) of Proposition 3.1, we obtain

$$\begin{aligned}
|(\Lambda(L_q y_{m-1}^q - y_{m-1}^q))(t)| &\leq \frac{1}{2}\alpha_1(t)\Big(\max_{s\in[a,b]}(L_q y_{m-1}^q(s) - y_{m-1}^q(s)) - \min_{s\in[a,b]}(L_q y_{m-1}^q(s) - y_{m-1}^q(s))\Big) \\
&\leq \alpha_1(t)\max_{s\in[a,b]}|L_q y_{m-1}^q(s) - y_{m-1}^q(s)| \\
&\leq \frac{1}{2}(b-a)\left(\frac{2}{\pi}\ln q + 1\right)E_q(y_{m-1}^q). \qquad (5.10)
\end{aligned}$$

Combining (5.8) with (5.10) and recalling (4.9), we find

$$\begin{aligned}
\left|v_m^q(t,\xi,\eta) - v_0^q(t,\xi,\eta)\right| &= (\Lambda L_q y_{m-1}^q)(t) \\
&= (\Lambda y_{m-1}^q)(t) + (\Lambda(L_q y_{m-1}^q - y_{m-1}^q))(t) \\
&\leq \frac{1}{4}(b-a)\left(\delta_{[a,b],\Omega_\varrho}(f) + 2\left(\frac{2}{\pi}\ln q + 1\right)E_q(y_{m-1}^q)\right). \qquad (5.11)
\end{aligned}$$

For $m = 1$, (5.11) and condition (4.11) yield

$$\begin{aligned}
\left|v_1^q(t,\xi,\eta) - v_0^q(t,\xi,\eta)\right| &\leq \frac{1}{4}(b-a)\left(\delta_{[a,b],\Omega_\varrho}(f) + 2\left(\frac{2}{\pi}\ln q + 1\right)E_q(N_f u_0(\cdot,\xi,\eta))\right) \\
&\leq \frac{1}{4}(b-a)\left(\delta_{[a,b],\Omega_\varrho}(f) + 2l_{q,\Omega_\varrho}(f)\right) \\
&\leq \varrho,
\end{aligned}$$

which, by virtue of (2.4), shows that (5.7) holds with $m = 1$. Arguing by induction, we show that (5.7) holds for any m. The values of every function of sequence (4.9) are thus contained in Ω_ϱ. Using the Lipschitz condition on f and Proposition 3.1, we get

$$\begin{aligned}
&|(N_f u_m(\cdot,\xi,\eta))(t) - (L_q N_f v_m^q(\cdot,\xi,\eta))(t)| \\
&\quad\leq |(N_f u_m(\cdot,\xi,\eta))(t) - (N_f v_m^q(\cdot,\xi,\eta))(t)| + |(N_f v_m^q(\cdot,\xi,\eta))(t) - (L_q N_f v_m^q(\cdot,\xi,\eta))(t)| \\
&\quad\leq K|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)| + \left(\frac{2}{\pi}\ln q + 1\right)E_q(N_f v_m^q(\cdot,\xi,\eta)) \qquad (5.12)
\end{aligned}$$

for all t and m.

Let us put

$$(My)(t) := \left(1 - \frac{t-a}{b-a}\right) \int_a^t y(s)ds + \frac{t-a}{b-a} \int_t^b y(s)ds, \qquad t \in [a,b], \tag{5.13}$$

for any continuous vector function y. Then, according to (4.1), (4.9), (4.6), and (5.12), we obtain

$$\begin{aligned} |u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)| &= |(\Lambda[N_f u_{m-1}(\cdot,\xi,\eta) - L_q N_f v_{m-1}^q(\cdot,\xi,\eta)])(t)| \\ &\le (M\,|N_f u_{m-1}(\cdot,\xi,\eta) - L_q N_f v_{m-1}^q(\cdot,\xi,\eta)\,|)(t) \\ &\le (MK|u_{m-1}(\cdot,\xi,\eta) - v_{m-1}^q(\cdot,\xi,\eta)\,|)(t) \\ &\quad + \left(\frac{2}{\pi}\ln q + 1\right) E_q(N_f v_m^q(\cdot,\xi,\eta))(Me)(t) \\ &\le (MK|u_{m-1}(\cdot,\xi,\eta) - v_{m-1}^q(\cdot,\xi,\eta)\,|)(t) + l_{q,\Omega_\varrho}(f)\,(Me)(t) \end{aligned}$$

for $t \in [a,b]$, $m \ge 1$, where $e = \mathrm{col}(1,1,\dots,1)$. In particular,

$$\begin{aligned} |u_1(t,\xi,\eta) - v_1^q(t,\xi,\eta)| &\le l_{q,\Omega_\varrho}(f)\,(Me)(t) \\ &= l_{q,\Omega_\varrho}(f)\alpha_1(t), \\ |u_2(t,\xi,\eta) - v_1^q(t,\xi,\eta)| &\le (MK|u_1(\cdot,\xi,\eta) - v_1^q(\cdot,\xi,\eta)\,|)(t) + l_{q,\Omega_\varrho}(f)\,(Me)(t) \\ &\le K(Ml_{q,\Omega_\varrho}\alpha_1 e)(t) + l_{q,\Omega_\varrho}(f)\alpha_1(t) \\ &= (K\alpha_2(t) + 1_n\alpha_1(t))\,l_{q,\Omega_\varrho}(f). \end{aligned}$$

Arguing by induction, we obtain

$$\left|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| \le (\alpha_m(t)K^{m-1} + \alpha_{m-1}(t)K^{m-2} + \dots + 1_n\alpha_1(t))\,l_{q,\Omega_\varrho}(f),$$

where α_k, $k = 1,2,\dots$, are given by (5.2), (5.3). Estimate (5.5) of Lemma 5.3 now yields

$$\left|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| \le \frac{10}{9}[1_n + K_* + K_*^2 + \dots K_*^{m-1}]\,\alpha_1(t)\,l_{q,\Omega_\varrho}(f)$$

with K_* as in (2.14), whence, due to assumption (2.8),

$$\left|u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| \le \frac{10}{9}(1_n - K_*)^{-1}\alpha_1(t)\,l_{q,\Omega_\varrho}(f). \tag{5.14}$$

Using (5.14) and estimate (2.13) of Theorem 2.1, we get

$$\begin{aligned} \left|u_\infty(t,\xi,\eta) - v_m^q(t,\xi,\eta)\right| &\le |(u_\infty(t,\xi,\eta) - u_m(t,\xi,\eta))| + |(u_m(t,\xi,\eta) - v_m^q(t,\xi,\eta))| \\ &\le \frac{10}{9}\alpha_1(t)\,K_*^m\,(1_n - K_*)^{-1}\,\delta_{[a,b],\Omega_\varrho}(f) \\ &\quad + \frac{10}{9}\,(1_n - K_*)^{-1}\,\alpha_1(t)\,l_{q,\Omega_\varrho}(f) \\ &= \frac{10}{9}\alpha_1(t)K_*^m(1_n - K_*)^{-1}(\delta_{[a,b],\Omega_\varrho}(f) + l_{q,\Omega_\varrho}(f)), \end{aligned} \tag{5.15}$$

where $u_\infty(\cdot,\xi,\eta)$ is a limit function (2.9) of sequence (2.2) (the limit exists by Theorem 2.1). In view of (2.8) and (2.14), estimate (5.15) shows that sequence (4.8), (4.9) converges to the same limit.

5.2 Proof of Theorem 4.3

We shall use the following Ostrowski inequality [4] for Lipschitz continuous functions [1].

Lemma 5.4 ([1]). *If $y : [a,b] \to \mathbb{R}$, $y \in H^1_c$, then*

$$\left| y(t) - \frac{1}{b-a}\int_a^b y(s)ds \right| \leq \left(\frac{1}{4} + \left(\frac{t - \frac{1}{2}(a+b)}{b-a} \right)^2 \right) c(b-a) \tag{5.16}$$

for all $t \in [a,b]$.

If $y \in H^1_c$, $y : [a,b] \to \mathbb{R}^n$, then c in (5.16) is a vector and the inequality is understood componentwise. Recall that H^1_c is the class of functions y satisfying (4.10) with $k_i = 1$, $i = 1,2,\dots,n$, i. e., y is Lipschitzian with the vector c.

In view of the observation made after the formulation of Theorem 4.3, we shall consider sequence (4.8), (4.9).

Fix $\xi \in D_0$ and $\eta \in D_1$ and write $v^q_m(t) = v^q_m(t,\xi,\eta)$ for the sake of brevity. Let us put

$$c^q_m := \max_{t\in[a,b]} |\dot{v}^q_m(t)|, \qquad m \geq 0,\ q \geq 1, \tag{5.17}$$

where $\dot{} = d/dt$. In other words, c^q_m is the Lipschitz constant of the polynomial v^q_m (we know from (4.9) that v^q_m is a polynomial of degree $\leq q+1$, i. e., $v^q_m \in \mathcal{P}_{q+1}$). Thus,

$$v^q_m \in H^1_{c^q_m}. \tag{5.18}$$

According to (4.2), (4.9), we have

$$\dot{v}^q_{m-1}(t) = \dot{u}_0(t) + (L_q N_f v^q_{m-2})(t) - \frac{1}{b-a}\int_a^b (L_q N_f v^q_{m-2})(s)ds. \tag{5.19}$$

Since, by (2.1),

$$\dot{u}_0(t) = \frac{1}{b-a}(\eta - \xi), \tag{5.20}$$

it follows from (5.19) and Lemma 5.4 that

$$|\dot{v}^q_{m-1}(t)| \leq \frac{1}{b-a}|\eta - \xi| + \left(\frac{1}{4} + \left(\frac{t - \frac{1}{2}(a+b)}{b-a} \right)^2 \right)(b-a)\lambda^q_{m-2}, \tag{5.21}$$

where λ^q_{m-2} is the Lipschitz constant (actually, vector) of the vector function $N_f v^q_{m-2}$.

By assumption, f satisfies condition (4.12) with $\beta = 1$. Therefore, by virtue of equality (5.17) and Lemma 5.1,

$$N_f v^q_{m-2} \in H^1_{Kc^q_{m-2}+c} \tag{5.22}$$

and, hence,

$$\lambda^q_{m-2} \leq Kc^q_{m-2} + c. \tag{5.23}$$

It is easy to check that

$$\max_{t\in[a,b]} (2t-a-b)^2 = (b-a)^2$$

and, therefore, combining (5.21) and (5.23), we obtain

$$\begin{aligned}|\dot{v}^q_{m-1}(t)| &\le \frac{1}{b-a}|\eta-\xi| + \frac{1}{4}\left(1+\left(\frac{2t-a-b}{b-a}\right)^2\right)(b-a)\lambda^q_{m-2}\\ &\le \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)\lambda^q_{m-2}\\ &\le \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)(Kc^q_{m-2}+c), \end{aligned} \tag{5.24}$$

whence, due to (5.17),

$$c^q_{m-1} \le \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)(Kc^q_{m-2}+c). \tag{5.25}$$

Using (5.25) and arguing by induction, we get

$$\begin{aligned}c^q_{m-1} \le h &+ \frac{1}{2}(b-a)Kh + \frac{1}{4}(b-a)^2K^2h + \frac{1}{8}(b-a)^3K^3h\\ &+ \cdots + \frac{1}{2^{m-2}}(b-a)^{m-2}K^{m-2}h + \frac{1}{2^{m-1}}(b-a)^{m-1}K^{m-1}c^q_0,\end{aligned} \tag{5.26}$$

where

$$h := \frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)c. \tag{5.27}$$

By (5.20), we have

$$c^q_0 = \frac{1}{2}|\eta-\xi|$$

and, therefore, (5.26) implies that

$$\begin{aligned}c^q_{m-1} &\le (1-K_0)^{-1}\left(\frac{1}{b-a}|\eta-\xi| + \frac{1}{2}(b-a)c\right) + \frac{1}{b-a}K_0^{m-1}|\eta-\xi|\\ &\le (1-K_0)^{-1}\left(\frac{1}{b-a}d + \frac{1}{2}(b-a)c\right) + \frac{1}{b-a}K_0^{m-1}d\\ &\le (1-K_0)^{-1}\left(\frac{1}{b-a}d + \frac{1}{2}(b-a)c\right) + \frac{1}{b-a}d,\end{aligned} \tag{5.28}$$

where

$$K_0 := \frac{1}{2}(b-a)K$$

and d is the vector defined componentwise as follows:

$$d := \operatorname{col}\left(\sup_{\xi\in D_0,\,\eta\in D_1}|\eta_1-\xi_1|, \sup_{\xi\in D_0,\,\eta\in D_1}|\eta_2-\xi_2|, \dots, \sup_{\xi\in D_0,\,\eta\in D_1}|\eta_n-\xi_n|\right).$$

Note that the term at the right-hand side of (5.28) depends neither on m nor on q.

Since λ^q_{m-1} denotes the Lipschitz constant of $N_f v^q_{m-1}$, it follows from Jackson's theorem (see [6, Corollary 1.4.2]) and inequality (5.23) that

$$\begin{aligned}E_q(N_f v^q_{m-1}) &\le \frac{6}{q}\lambda^q_{m-1}(b-a)\\ &\le \frac{6}{q}(Kc^q_{m-1}+c)(b-a),\end{aligned} \tag{5.29}$$

whence, using (5.28), we obtain

$$E_q(N_f v_{m-1}^q) \le \frac{6}{q}\Big(K(1-K_0)^{-1}\Big(\frac{1}{b-a}d + \frac{1}{2}(b-a)c\Big) + \frac{1}{b-a}d\Big)(b-a)$$
$$= \frac{6}{q}\Big(K(1-K_0)^{-1}\Big(d + \frac{1}{2}(b-a)^2 c\Big) + d\Big). \tag{5.30}$$

Recall that we use notation (4.5) for vector functions and the inequalities in (5.29), (5.30) are componentwise.

Estimate (5.30) implies that, by choosing $q_m = q$, $m \ge 1$, with q large enough, we guarantee the fulfilment of condition (4.21), which, as have already been said, ensures the converegence of sequence (4.19), (4.20), or, which is the same in this case, of sequence (4.8), (4.9).

6 A numerical example

Let us apply the approach described above to the system of differential equations with transcendental non-linearities

$$\begin{aligned} u_1'(t) &= u_1(t)u_2(t), \\ u_2'(t) &= -\ln(2u_1(t)), \qquad t \in [0, \pi/4], \end{aligned} \tag{6.1}$$

considered under the non-linear two-point boundary conditions

$$(u_1(a))^2 + (u_2(b))^2 = \frac{3}{8}, \qquad u_1(a)u_2(b) = \frac{\sqrt{2}}{8}. \tag{6.2}$$

We have $a = 0$, $b = \pi/4$, $f = \mathrm{col}(f_1, f_2)$,

$$f_1(t, u_1, u_2) = u_1 u_2, \qquad f_2(t, u_1, u_2) = -\ln(2u_1) \tag{6.3}$$

and $\phi(u) = \mathrm{col}((u_1(a))^2 + (u_2(b))^2 - 3/8,\ u_1(a)u_2(b) - \sqrt{2}/8)$ in this case.

Introduce the vectors of parameters $\xi = \mathrm{col}(\xi_1, \xi_2)$, $\eta = \mathrm{col}(\eta_1, \eta_2)$ and, instead of problem (6.1), (6.2), consider (6.1) under the parametrised boundary conditions (1.3).

Let us choose the sets D_0 and D_1, where one looks the values $u(a)$ and $u(b)$, e.g., as follows:

$$D_0 = \{(u_1, u_2) : 0.35 \le u_1 \le 0.75,\ 0.35 \le u_2 \le 0.55\}, \qquad D_1 = D_0. \tag{6.4}$$

Note that this choice of sets is motivated by the results of computation (it is always useful to start the computation before trying to check the conditions in order to avoid unnecessary computations, see Section 6.1).

According to (2.3), it follows from (6.4) that $\Omega = D_0$. For $\varrho = \mathrm{col}(\varrho_1, \varrho_2)$, we choose the value

$$\varrho = \mathrm{col}(0.2, 0.4). \tag{6.5}$$

Then, in view of (6.4), (6.5), set (2.4) has the form

$$\Omega_\varrho = \{(u_1, u_2) : 0.15 \le u_1 \le 0.95,\ -0.05 \le u_2 \le 0.95\}. \tag{6.6}$$

According to (2.6), (6.3), and (6.6),

$$\frac{b-a}{4}\delta_{[a,b],\Omega_\varrho}(f) = \frac{\pi}{8}\Big(\max_{(t,u)\in[a,b]\times\Omega_\varrho} f(t,u) - \inf_{(t,u)\in[a,b]\times\Omega_\varrho} f(t,u)\Big)$$
$$\approx \frac{\pi}{8}\begin{pmatrix}0.95\\1.845826690\end{pmatrix} \approx \begin{pmatrix}0.1865320638\\0.3624272230\end{pmatrix} < \begin{pmatrix}0.2\\0.4\end{pmatrix} = \varrho, \tag{6.7}$$

which means that, for ϱ given by (6.5), condition (4.18) holds with $r_1 < 0.013$, $r_2 < 0.037$. Then, by Proposition 4.2, the scheme (4.19), (4.20) is applicable for sufficiently large numbers of nodes if f is Lipschitzian on Ω_ϱ with a matrix K satisfying condition (2.8). However, a direct computation shows that $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with

$$K = \begin{pmatrix}0.95 & 0.95\\6.7 & 0\end{pmatrix}, \tag{6.8}$$

whence, after determining the eigenvalues, we find that (2.8) is satisfied:

$$r(K) \approx 3.04222 < 4.24413 \approx \frac{40}{3\pi} = \frac{10}{3(b-a)}.$$

We can now proceed to the construction of approximations. The question on choosing a suitable value of q we will treat in a heuristic manner and select a certain value according to the practical experience; for larger, "guaranteed" values of q, the quality of results still increases.

We thus use the iteration process $\{v_m^q(\cdot,\xi,\eta) : m \geq 0\}$ defined according to equalities (4.8), (4.9). Using Maple 17, we carry out computations for several values of m at different numbers of Chebyshev nodes on the interval $[a,b]$.

6.1 Approximations of the first solution

It is easy to verify by substitution that

$$u_1^*(t) = \frac{1}{2}\exp\Big(\frac{1}{2}\sin t\Big), \qquad u_2^*(t) = \frac{1}{2}\cos t \tag{6.9}$$

is a solution of problem (6.1), (6.2). Let us show how the corresponding approximate solutions are constructed according to the method indicated above.

Putting, e. g., $q = 4$, we get the corresponding five Chebyshev nodes (3.3) transformed from $(-1,1)$ into interval (a,b):

$$t_1 = 0.7661781024, \qquad t_2 = 0.6235218106, \qquad t_3 = 0.3926990817,$$
$$t_4 = 0.1618763528, \qquad t_5 = 0.0192200611.$$

The approximate determining system (4.15), (4.16), by solving which the numerical values of the parameters determining the approximate solutions are obtained, for this example is constituted by four scalar non-linear equations with respect to ξ_1, ξ_2, η_1, η_2. For $m = 0$, it has the form

$$\begin{aligned} \eta_1 - \xi_1 &= 0.2617993878\,\eta_1\eta_2 + 0.1308996940\,\eta_1\xi_2 + 0.1308996940\,\xi_1\eta_2 \\ &\quad + 0.2617993878\,\xi_1\xi_2, \\ \eta_2 - \xi_2 &= -0.20638381\,\ln(0.4122147477\,\eta_1 + 1.587785252\,\xi_1) \\ &\quad - 0.20638383\,\ln(1.587785252\,\eta_1 + 0.4122147477\,\xi_1) \\ &\quad - 0.065887535\,\ln(0.0489434837\,\eta_1 + 1.951056516\,\xi_1) \\ &\quad - 0.065887536\,\ln(1.951056516\,\eta_1 + 0.0489434837\,\xi_1) \\ &\quad - 0.24085543\,\ln(\xi_1 + \eta_1), \\ \xi_1\eta_2 &= 0.1767766952, \\ \eta_2^2 + \xi_1^2 &= 0.375. \end{aligned} \tag{6.10}$$

Solving (6.10) for $\xi_1 \in (0.45, 0.55)$, we get the root

$$\xi_1 = 0.5000000003, \quad \xi_2 = 0.4910030682, \quad \eta_1 = 0.6966729228, \quad \eta_2 = 0.3535533902, \tag{6.11}$$

by substituting which into formula (4.8) the *zeroth approximation* $U_0 = \mathrm{col}(U_{01}, U_{02})$ (i. e., function (4.17) for $m = 0$) is obtained:

$$U_{01}(t) = 0.5000000003 + 0.2504117432t, \qquad U_{02}(t) = 0.4910030705 - 0.17500063683t. \tag{6.12}$$

This initial approximation is obtained before any iteration is carried out and is useful as a source of preliminary information on the localisation of solutions (in particular, the graph of function (6.12) is a motivation to choose D_0, D_1 in form (6.4)).

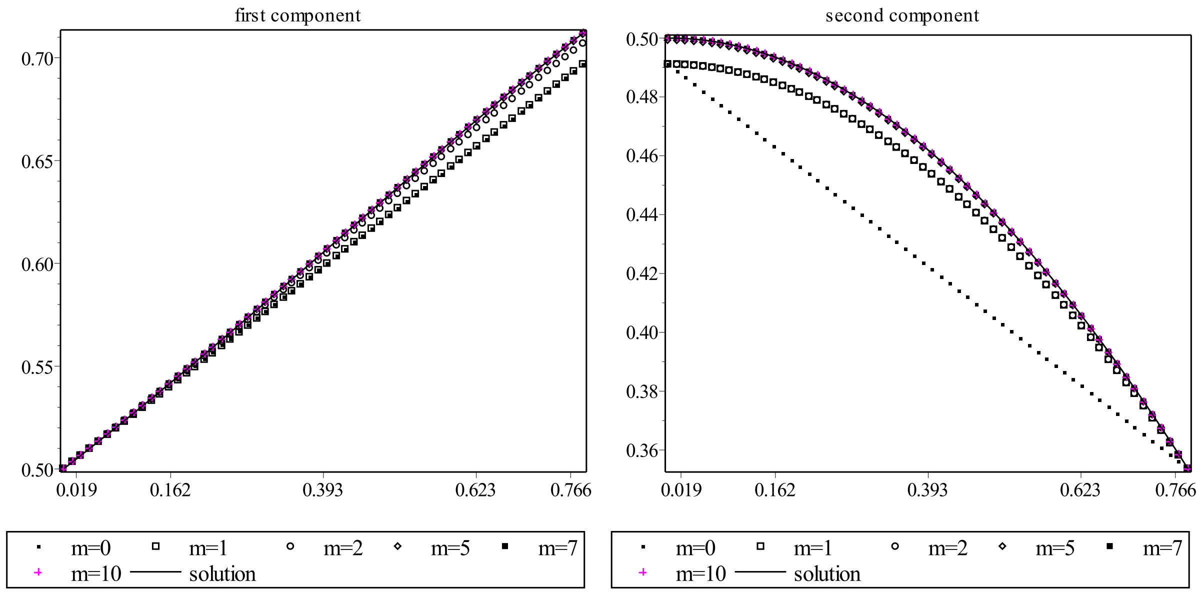
In order to construct higher approximations, we use the *frozen parameters* simplification [14], i. e., before passing from step m to step $m+1$, we substitute the roots of the mth approximate determining equation into the formula obtained on step m. In this way, at the expense of some extra error which tends to zero as m grows, the construction of determining equations is considerably simplified. Note also that, at every step of iteration carried out according to (4.8), (4.9), we obtain a polynomial of degree $\leq q+1$.

Constructing the functions $v_m^4(\cdot, \xi, \eta)$ for several values of m and solving the corresponding approximate determining systems (4.15), (4.16), we obtain the numerical values of the parameters presented in Table 6.1. The last row of the table contains the exact values corresponding to solution (6.9). Since $q = 4$, all these approximations are polynomials of degree 5; e. g., for $m = 7$, it has the form

$$U_{71}^4(t) \approx 0.00456\,t^5 - 0.02668\,t^4 - 0.02838\,t^3 + 0.06195\,t^2 + 0.24987\,t + 0.5, \tag{6.13}$$

$$U_{72}^4(t) \approx 0.49982 - 0.0017\,t^5 + 0.02231\,t^4 - 0.00062\,t^3 - 0.24956\,t^2 + 0.49982. \tag{6.14}$$

The graphs of the seventh approximation (6.13), (6.14) and of the exact solution (6.9) are shown on Figure 6.1.



(a) (b)

Figure 6.1: First solution: $q = 4$, $m = 0, 1, 2, 5, 7, 10$.

m	ξ_1	ξ_2	η_1	η_2
0	0.5000000003	0.4910030682	0.6966729228	0.3535533902
1	0.5000000003	0.4910030705	0.6966729234	0.3535533902
2	0.5000000003	0.4909073352	0.7067944705	0.3535533902
5	0.5000000003	0.4990243859	0.7110836712	0.3535533902
7	0.5000000003	0.4997040346	0.7117894333	0.3535533902
10	0.5000000003	0.4999499916	0.7120202126	0.3535533902
16	0.5000000003	0.4999983385	0.7120583725	0.3535533902
20	0.5000000003	0.4999993608	0.7120592079	0.3535533902
	....	....	....	....
∞	$\frac{1}{2}$	$\frac{1}{2}$	0.7120595095	0.3535533905

Table 6.1: First solution: values of parameters for $q = 4$.

m	ξ_1	ξ_2	η_1	η_2
0	0.5000000003	0.4910340532	0.696681237	0.3535533902
1	0.5000000003	0.4909136731	0.7068092824	0.3535533902
2	0.5000000003	0.4969678528	0.7084223215,	0.3535533902
3	0.5000000003	0.4975642896	0.7104804038	0.3535533902
4	0.5000000003	0.4990270554	0.7110851380	0..3535533902
5	0.50000000039	0.4993503524	0.711592909	0.3535533902
6	0.5000000003	0.4997051246	0.7117900289	0.3535533902
7	0.5000000003	0.4998223937	0.7119239307	0.3535533902
	....	....	....	....
∞	$\frac{1}{2}$	$\frac{1}{2}$	0.7120595095	0.3535533905

Table 6.2: First solution: values of parameters for $q = 11$.

m	ξ_1	ξ_2	η_1	η_2
11	0.4999999999	0.4999103564	0.7120195453	0.3535533905
12	0.4999999999	0.4999501433	0.7120195453	0.3535533905

Table 6.3: First solution: values of parameters for $q = 17$.

For $q = 11$, the Chebyshev nodes (3.3) on (a, b) have the form

$$\begin{aligned} &t_1 = 0.7820385685, && t_2 = 0.7555057258, && t_3 = 0.70424821007, && t_4 = 0.6317591359,\\ &t_5 = 0.5429785144, && t_6 = 0.4439565976, && t_7 = 0.3414415658, && t_8 = 0.242419649,\\ &t_9 = 0.1536390274, && t_{10} = 0.0811499534, && t_{11} = 0.0298924377, && t_{12} = 0.0033595951. \end{aligned}$$

Computing several approximations, we get from (4.15), (4.16) the numerical values for the parameters presented in Table 6.2. Table 6.3 contains the approximate values of parameters for $q = 17$ and $m \in \{11, 12\}$.

6.2 Approximations of the second solution

Choosing different constraints when solving the approximate determining system (4.15), (4.16), we find that, along with the root from Table 6.1, it has also another root presented in Table 6.4. It is quite evident from the results of computation that this indicates the existence of another solution of the boundary value problem (6.1), (6.2), which is different from (6.9).

On Figure 6.2, one can see the graph of approximations to the second solution, while Figure 6.3 shows the residuals obtained by substituting these approximations into the given differential system (i.e., the functions $t \mapsto U'_{mk}(t) - f_k(t, U_m(t))$, $k = 1, 2$). We see that, e.g., at $m = 10$, we get a residual of order about 10^{-5}. The computation of 20 approximations with $q = 4$ on a standard portable computer with Intel® Core i3-2310M CPU @ 2.10 GHz takes about 130 seconds.

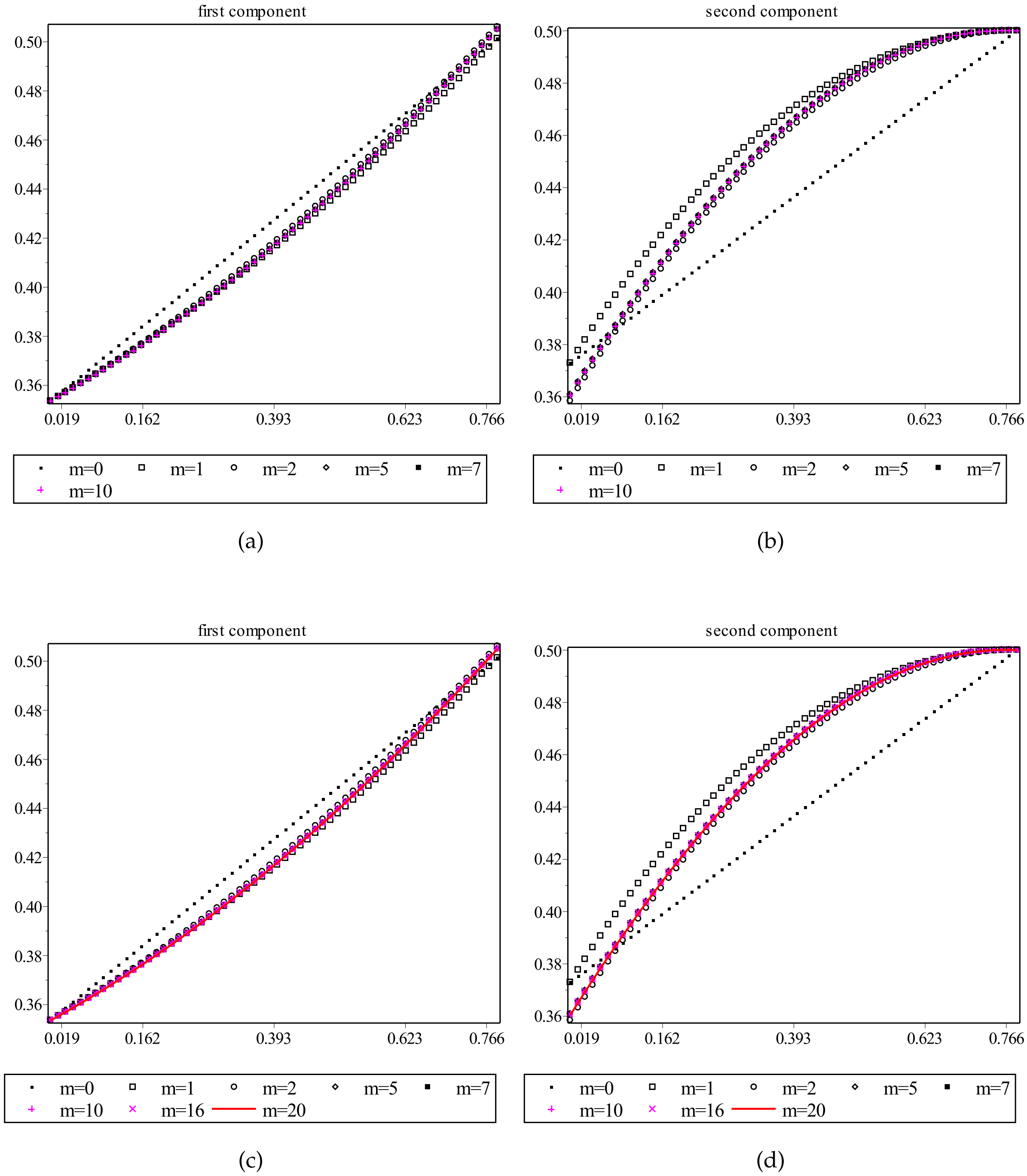


Figure 6.2: Second solution: $q = 4$, $m = 0, 1, 2, 5, 7, 10, 16, 20$.

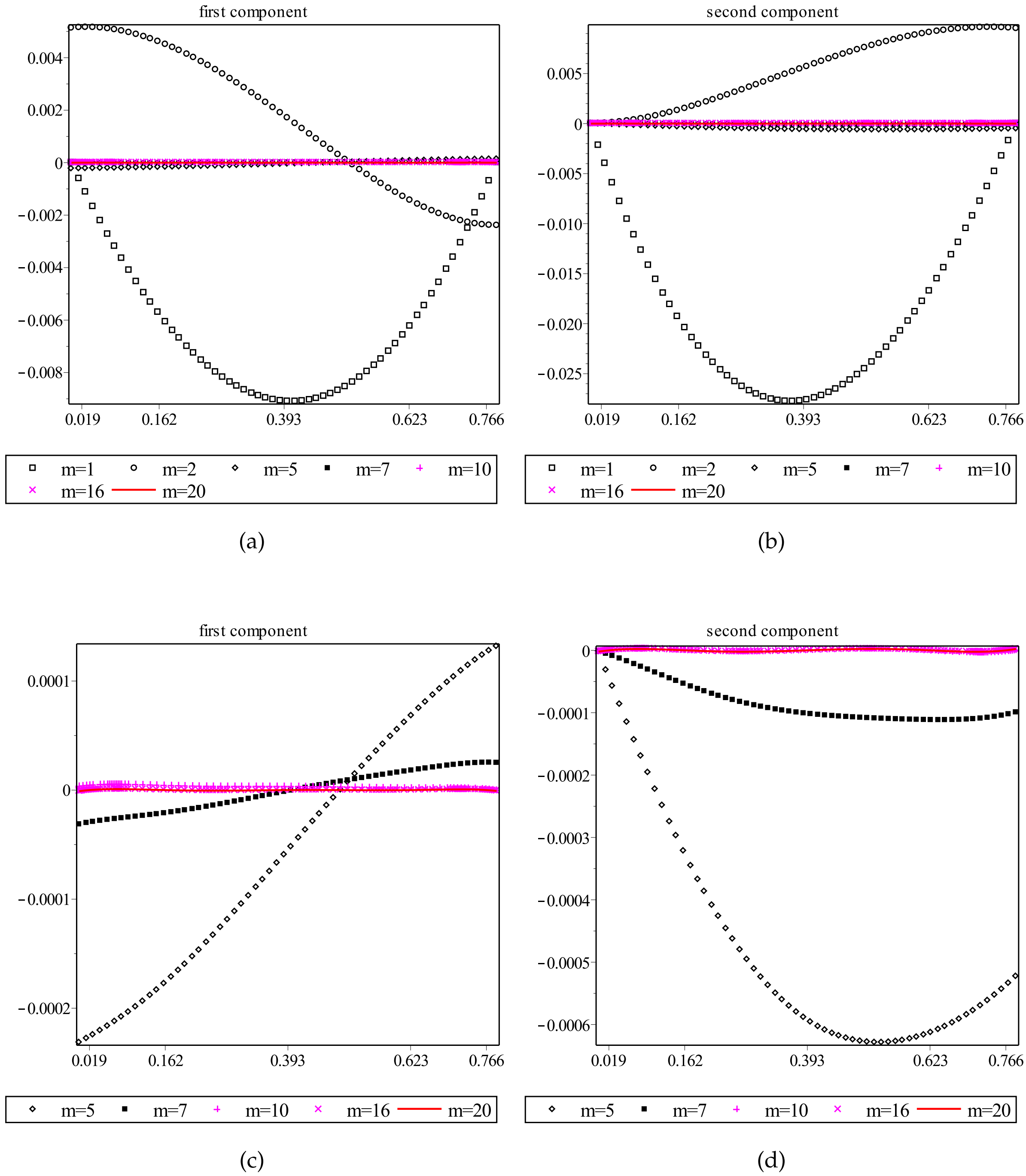


Figure 6.3: The residuals of approximations to the second solution: $q = 4$, $m = 1, 2, 5, 7, 10, 16, 20$.

m	ξ_1	ξ_2	η_1	η_2
0	0.3535533902	0.372879209	0.5012944951	0.5000000003
1	0.3535533902	0.372879209	0.5012944951	0.5000000003
2	0.3535533902	0.3583701009	0.5060832907	0.5000000003
5	0.3535533902	0.360895369	0.5049836277	0.5000000002
7	0.3535533902	0.3606070436	0.5049746944	0.5000000003
10	0.3535533902	0.3605371997	0.504964082	0.5000000003
16	0.3535533902	0.3605333927	0.5049600567	0.5000000003
20	0.3535533902	0.3605332714	0.5049599787	0.5000000003

Table 6.4: Second solution: values of parameters for $q = 4$.

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